Homotopy-commutativity in spinor groups

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1 Introduction

For two subsets $S$ and $S'$ of a topological group $G$ which contain the unit of $G$ as its base points, we say $S$ and $S'$ homotopy-commute in $G$, when the commutator map $c$ from $S \wedge S'$ to $G$ which maps $(x, y) \in S \wedge S'$ to $xyx^{-1}y^{-1} \in G$ is null homotopic.

In [3], the first author showed the next theorem:

Theorem 1.1. Let $n$, $m$ be positive integers, and let $n + m \neq 4$ or $8$. If $n$ or $m$ is even or if $\binom{n+m-2}{n-1} \equiv 0 \mod 2$ then $SO(n)$ and $SO(m)$ do not homotopy-commute in $SO(n + m - 1)$.

In this paper, we describe the homotopy-commutativity of $Spin(n)$ and $Spin(m)$ in $Spin(n + m - 1)$.

Definition 1.2. If $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n + m - 1)$, we say $(n, m)$ is SO-irregular, and if not we say $(n, m)$ is SO-regular. Also, If $Spin(n)$ and $Spin(m)$ homotopy-commute in $Spin(n + m - 1)$, we say $(n, m)$ is Spin-irregular, and if not we say $(n, m)$ is Spin-regular.
Main theorems are the followings:

**Theorem 1.3.** Assume neither \( n - 1 \) nor \( m - 1 \) is a power of 2 and \( n + m \neq 4 \) or 8. If \( n \) or \( m \) is even or if \((n + m - 1) \equiv 0 \mod 2\) then \((n, m)\) is Spin-regular.

For the case \( n - 1 \) is a power of 2, we give some results as following:

**Theorem 1.4.** Set \( n = 3 \) and \( m \equiv 1 \mod 4 \) then \((3, m)\) is Spin-irregular.

**Remark 1.5.** Theorem 1.1 implies that if \( m \not\equiv 1 \mod 4 \), \((3, m)\) is SO-regular.

**Remark 1.6.** In fact, since \(\text{Spin}(5) \cong Sp(2)\) and \(\pi_6(Sp(2)) \cong \pi_6(Sp) \cong \tilde{KSp}^{-7}(\text{pt}) \cong 0\) where \(Sp\) is \(\lim_{n \to \infty} Sp(n)\), the commutator map \(c : \text{Spin}(3) \wedge \text{Spin}(3) \to \text{Spin}(5)\) is null homotopic and \((3, 3)\) is Spin-irregular. On the other hand, Theorem 1.1 implies \((3, 3)\) is SO-regular. Therefore SO-regularity and Spin-regularity is not the same.

This paper is organized as follows: In §2 we give a sufficient condition for \((n, m)\) to be Spin-regular, which is an existence of a map with an adequate property and show that when \(n + m\) is odd, \((n, m)\) is Spin-regular. In §3 we introduce the maps \(\phi_{i,j} : \Omega^iBO \wedge \Omega^jBO \to \Omega^{i+j}BO\) to investigate \(\tilde{K}\tilde{O}^{-*}(\text{Spin}(n) \wedge \text{Spin}(m))\) and in §4 investigate its induced cohomology maps and prove Theorem 1.3 for the case both \(n\) and \(m\) are odd. In §5 we look into the case \(n\) and \(m\) are even and complete the proof of Theorem 1.3 and finally in §6 we give the proof of Theorem 1.4.

## 2 Lift of commutator map

Similarly to [3], consider the next fibrations:

\[
\begin{align*}
\text{Spin}(n + m - 1) & \xrightarrow{i} \text{Spin} \xrightarrow{p} \text{Spin}/\text{Spin}(n + m - 1), \\
\text{SO}(n + m - 1) & \xrightarrow{j} \text{SO} \xrightarrow{q} \text{SO}/\text{SO}(n + m - 1),
\end{align*}
\]

where \(\text{SO}\) (resp. \(\text{Spin}\)) is \(\lim_{n \to \infty} SO(n)\) (resp. \(\lim_{n \to \infty} \text{Spin}(n)\)).

We refer to the cohomology rings of spaces which we use in this paper, that is,

\[
\begin{align*}
H^*(\Omega\text{Spin}) &= \mathbb{Z}/2\mathbb{Z}[\alpha_2, \alpha_4, \alpha_6, \cdots]/(\alpha_{4k} - \alpha_{2k}^2), \\
H^*(\text{Spin}(k)/\text{Spin}(k - l)) &= \Delta(x_{k-1}, \cdots, x_{k-1}), \\
H^*(\text{Spin}(k)) &= \Delta(x_3, x_5, x_7, x_9, \cdots) \otimes \bigwedge(z).
\end{align*}
\]
In the last equality, the index $i$ of $x_i$ scans all integers neither of which is not a power of 2 and $3 \leq i \leq k - 1$. Also, $\deg(\alpha_2) = 2i$ and $\deg(x_i) = i$.

Further, it can be easily seen that $H^*(\Omega Spin/\Spin(n + m - 1)) = 0$ for $* \leq n + m - 3$ and $H^{n+m-2}(\Omega Spin/\Spin(n + m - 1)) = \mathbb{Z}/2\mathbb{Z}$ whose generator is written as $\alpha_{n+m-2}$.

When $n + m$ is even, $\Omega^*(\alpha_{n+m-2}) = \alpha_{n+m-2} \in H^*(\Omega Spin)$.

From above fibrations, we can deduce the following fibration sequences.

$$
\begin{aligned}
\cdots & \to \Omega Spin \xrightarrow{\Omega p} \Omega(\Spin/\Spin(n + m - 1)) \xrightarrow{\delta_{\Spin}} \\
& \quad Spin(n + m - 1) \xrightarrow{i} Spin/\Spin(n + m - 1), \\
\cdots & \to \Omega SO \xrightarrow{\Omega q} \Omega(SO/\SO(n + m - 1)) \xrightarrow{\delta_{\SO}} \\
& \quad SO(n + m - 1) \xrightarrow{j} SO/\SO(n + m - 1).
\end{aligned}
$$

Let $c_{SO}$ (resp. $c_{\Spin}$) be the commutator map from $SO(n) \wedge SO(m)$ to $SO(n + m - 1)$ (resp. from $\Spin(n) \wedge \Spin(m)$ to $\Spin(n + m - 1)$). Obviously we can see that $i \circ c_{\Spin}$ and $j \circ c_{SO}$ are null homotopic. Thus there exists a lift of $c_{SO}$ from $SO(n) \wedge SO(m)$ to $\Omega SO/\SO(n + m - 1)$ and a lift of $c_{\Spin}$ from $\Spin(n) \wedge \Spin(m)$ to $\Omega Spin/\Spin(n + m - 1)$.

In [4], a lift of $c_{SO}$ written as $\lambda_{SO}$ was constructed and in [3], it is obtained that

$$
\lambda_{SO}^*(\alpha_{n+m-2}) = x_{n-1} \otimes x_{m-1}.
$$

Here set $\lambda_{\Spin} = \lambda_{SO} \circ (p_n \wedge p_m)$.

**Lemma 2.1.** $\lambda_{\Spin}$ is a lift of $c_{\Spin}$, that is, $\delta_{\Spin} \circ \lambda_{\Spin} \simeq c_{\Spin}$.

**Proof.** See the diagram below.
Since $\delta_{SO} \circ \lambda_{SO} \simeq c_{SO}$ and $\delta_{SO} \simeq p_{n+m-1} \circ \delta_{Spin}$, it occurs that

$$p_{n+m-1} \circ \delta_{Spin} \circ \lambda_{Spin} = \delta_{SO} \circ \lambda_{SO} \circ (p_n \land p_m) \simeq c_{SO} \circ (p_n \land p_m) = p_{n+m-1} \circ c_{Spin}$$  \hspace{1cm} (2)

Now consider the fibration $\mathbb{Z}/2\mathbb{Z} \to Spin(n + m - 1) \to SO(n + m - 1)$. Then for any CW complex $X$ we have the exact sequence of base pointed homotopy sets:

$$[X, \mathbb{Z}/2\mathbb{Z}]_* \to [X, Spin(n + m - 1)]_*^{p_{n+m-1} *}[X, SO(n + m - 1)]_*$$

Thus $p_{n+m-1 *}$ is injective and from 2 we can see

$$\delta_{Spin} \circ \lambda_{Spin} \simeq c_{Spin}.$$  \hspace{1cm} Q.E.D.

In the rest of paper, $c, \lambda, \delta$ stands for $c_{Spin}, \lambda_{Spin}, \delta_{Spin}$ respectively.

**Proposition 2.2.** Assume neither $n - 1$ nor $m - 1$ is a power of 2.

1. If $n + m$ is odd, $c$ is not null homotopic.

2. Let $n + m$ is even. If for any continuous map $x$ from $Spin(n) \land Spin(m)$ to $\Omega Spin$, $x^*(\alpha_{n+m-2}) \neq x_{n-1} \otimes x_{m-1}$ in cohomology, then $c$ is not null homotopic.

**Proof.**

If $c$ is null homotopic, that is, $\delta \circ \lambda \simeq *$, then there exists a map $x : Spin(n) \land Spin(m) \to \Omega Spin$ such that $\Omega p \circ x \simeq \lambda$.

From (1) we can see

$$x^*(\alpha_{n+m-2}) = x^* \circ \Omega p^*(\alpha_{n+m-2}) = \lambda^*(\alpha_{n+m-2}) = (p_n \land p_m)^* \circ \lambda_{SO}^*(\alpha_{n+m-2}) = (p_n \land p_m)^*(x_{n-1} \otimes x_{m-1}) = x_{n-1} \otimes x_{m-1},$$  \hspace{1cm} (3)

since neither $n - 1$ nor $m - 1$ is a power of 2. Thus the statement for the case $n + m$ is even is proved.
When \( n + m \) is odd, it occurs that
\[
\lambda^*(\alpha_{n+m-2}) = x^* \circ \Omega p^*(\alpha_{n+m-2}) = x^*(0),
\]
since \( H^*(\Omega \text{Spin}) \) is concentrated in even degrees. This contradicts to (3) and \( c \) is not null homotopic.

\[ \text{Q.E.D.} \]

3 \( \widetilde{KO}^{-*}(\text{Spin}(n) \wedge \text{Spin}(m)) \)

In this section we assume that both \( n \) and \( m \) are odd.

From Proposition 2.2 we should look into the homotopy set \([\text{Spin}(n) \wedge \text{Spin}(m), \Omega \text{Spin}]\). By use of KO-theory we can say that,
\[
[\text{Spin}(n) \wedge \text{Spin}(m), \Omega \text{Spin}] \cong [\text{Spin}(n) \wedge \text{Spin}(m), \Omega_0 \text{SO}] \cong \widetilde{KO}^{-2}(\text{Spin}(n) \wedge \text{Spin}(m)),
\]
since \( \Omega^2 \text{BO} \cong \Omega \text{SO} \).

Further more, the complex representation ring of \( \text{Spin}(2k+1) \) is generated by real representations or symplectic representations. (See Proposition 6.19 in P.290 of [8].) Thus Theorem 5.12. in [11] implies that, when \( n \) is odd, \( KO^{-*}(\text{Spin}(n)) \) is \( KO^{-*}(\text{pt}) \) free. Therefore we have a decomposition of
\[
\widetilde{KO}^{-*}(\text{Spin}(n) \wedge \text{Spin}(m)) \cong \widetilde{KO}^{-*}(\text{Spin}(n)) \otimes \frac{\widetilde{KO}^{-*}(\text{Spin}(m))}{\widetilde{KO}^{-*}(\text{pt})}.
\]

From now on, we identify \( \widetilde{KO}^{-1}(X) \) with \([X, \Omega^i \text{BO}]\).

**Theorem 3.1.** There is a map \( \phi_{i,j} : \Omega^i \text{BO} \wedge \Omega^j \text{BO} \to \Omega^{i+j} \text{BO} \) such that for any CW-complexes \( X, X' \) and \( \alpha \in \widetilde{KO}^{-i}(X) \) and \( \beta \in \widetilde{KO}^{-j}(X') \),
\[
\alpha \hat{\otimes} \beta = \phi_{i,j} \circ (\alpha \wedge \beta) \quad \text{in} \quad \widetilde{KO}^{-(i+j)}(X \wedge X').
\]

**Proof.** First we construct \( \phi_{i,j} \). Let \( \xi_n \) be the universal vector bundle over \( \text{BO}(n) \) and put \( \eta_n = \xi_n - n, \eta_\infty = \lim_{n \to \infty} \eta_n \). And set \( \phi_{0,0} : \text{BO} \wedge \text{BO} \to \text{BO} \) as the classifying map of \( \eta_\infty \hat{\otimes} \eta_\infty \). Let \( \kappa_i : \Sigma^i \Omega^i \text{BO} \to \text{BO} \) be the map which satisfies
\[
\text{Ad}^i \kappa_i \simeq \text{Id}_{\Omega^i \text{BO}}.
\]
Consider the composition of $\kappa_i \land \kappa_j$ and $\phi_{0,0}$:

$$\Sigma^i \Omega^j BO \land \Sigma^j \Omega^i BO \xrightarrow{\kappa_i \land \kappa_j} BO \land BO \xrightarrow{\phi_{0,0}} BO.$$ 

We define $\phi_{i,j}$ as

$$\phi_{i,j} = Ad^{i+j}(\phi_{0,0} \circ (\kappa_i \land \kappa_j)) : \Omega^i BO \land \Omega^j BO \longrightarrow \Omega^{i+j} BO.$$ 

Now, take $\alpha \in [X, \Omega^i BO]$ and $\beta \in [X', \Omega^j BO]$ and see the composition of $\alpha \land \beta$ and $\phi_{i,j}$:

$$\phi_{i,j} \circ (\alpha \land \beta) : X \land X' \longrightarrow \Omega^i BO \land \Omega^j BO \longrightarrow \Omega^{i+j} BO.$$ 

Taking $Ad^{-i-j}(\phi_{i,j})$ of the above composition, we obtain

$$Ad^{-i-j}(\phi_{i,j} \circ (\alpha \land \beta)) = (Ad^{-i-j}(\phi_{i,j}) \circ (\Sigma^i \alpha \land \Sigma^j \beta)) : \Sigma^{i+j}(X \land X') \longrightarrow \Sigma^{i+j}(\Omega^i BO \land \Omega^j BO) \rightarrow BO.$$ 

From definition of $\phi_{i,j}$, $Ad^{-i-j}(\phi_{i,j} \circ (\alpha \land \beta))$ is the composition of following maps:

$$\Sigma^{i+j}(X \land X') \xrightarrow{\Sigma^i \alpha \land \Sigma^j \beta} \Sigma^{i+j}(\Omega^i BO \land \Omega^j BO) \xrightarrow{\kappa_i \land \kappa_j} BO \land BO \xrightarrow{\phi_{0,0}} BO. \quad (4)$$

**Lemma 3.2.** For any continuous map $f : \Sigma^i X \longrightarrow BO$,

$$f \simeq \kappa_i(\Sigma^i Ad^i f).$$

**Proof.** Consider the composition of $Ad^i f$ and identity map of $\Omega^i BO$.

$$X \xrightarrow{Ad^i f} \Omega^i BO \xrightarrow{\text{Id}_{\Omega^i BO}} \Omega^i BO.$$ 

Taking $Ad^{-i}$ of the above composition, we have

$$f = Ad^{-i}(\text{Id}_{\Omega^i BO} \circ Ad^i f) = \kappa_i \circ \Sigma^i Ad^i f$$

$$: \Sigma^i X \xrightarrow{\Sigma^i Ad^i f} \Sigma^i \Omega^i BO \xrightarrow{\kappa_i} BO.$$ 

Q.E.D.

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By (4) and the above lemma, it follows that
\[
\operatorname{Ad}^{-(i+j)}(\phi_{i,j} \circ (\alpha \land \beta)) \simeq \phi_{0,0} \circ (\kappa_i \land \kappa_j) \circ (\Sigma^i \alpha \land \Sigma^j \beta) \\
\simeq \phi_{0,0} \circ (\kappa_i \circ \Sigma^i \alpha) \land (\kappa_j \circ \Sigma^j \beta) \\
\simeq \phi_{0,0} \circ (\operatorname{Ad}^{-i} \alpha \land \operatorname{Ad}^{-j} \beta).
\]

Since \( f \in [X, \Omega^i BO] \) corresponds to \( (\operatorname{Ad}^{-i} f)^\ast(\eta_\infty) \in \widetilde{KO}^{-i}(X) \), the above equation says that \( \phi_{i,j} \circ (\alpha \land \beta) \) corresponds to
\[
(\operatorname{Ad}^{-i} \alpha \land \operatorname{Ad}^{-j} \beta)^\ast(\eta_\infty) = \operatorname{Ad}^{-i} \alpha^\ast(\eta_\infty) \hat{\otimes} \operatorname{Ad}^{-j} \beta^\ast(\eta_\infty).
\]
Therefore we obtain that
\[
\alpha \hat{\otimes} \beta = \phi_{i,j} \circ (\alpha \land \beta) \quad \text{in} \quad \widetilde{KO}^{-(i+j)}(X \land X').
\]
Q.E.D.

From the above theorem, we can deduce the next theorem.

**Theorem 3.3.** Assume both \( n \) and \( m \) are odd. If, for all \((i, j) \in \mathbb{Z}/8\mathbb{Z}^2 \) which satisfy \( i + j = 2 \), \( \phi_{i,j}^\ast(\alpha_{n+m-2}) = \sum b_s \otimes c_t \) where \( |b_s| = s \) and \( |c_t| = t \) and \( b_{n-1} \otimes c_{m-1} = 0 \) then \( c : \text{Spin}(n) \land \text{Spin}(m) \to \text{Spin}(n + m - 1) \) is not null homotopic.

**Proof.** For any \( \eta \in \widetilde{KO}^{-2}(\text{Spin}(n) \land \text{Spin}(m)) \), there exist \( \alpha_a \in \widetilde{KO}^{-i_a}(\text{Spin}(n)) \) and \( \beta_a \in \widetilde{KO}^{-j_a}(\text{Spin}(m)) \) such that \( \eta = \sum \alpha_a \hat{\otimes} \beta_a \) and \( i_a + j_a = 2 \). Since \( \alpha_{n+m-2} \) is primitive,
\[
\eta^\ast(\alpha_{n+m-2}) = (\sum \alpha_a \hat{\otimes} \beta_a)^\ast(\alpha_{n+m-2}) = \sum (\alpha_a \hat{\otimes} \beta_a)^\ast(\alpha_{n+m-2})
\]
and by Theorem 3.1,
\[
(\alpha \hat{\otimes} \beta)^\ast(\alpha_{n+m-2}) = (\alpha \land \beta)^\ast \circ \phi_{i,j}^\ast(\alpha_{n+m-2}).
\]
If the hypothesis is satisfied, \( \eta^\ast(\alpha_{n+m-2}) \) can not be \( x_{n-1} \otimes x_{m-1} \). Therefore from Proposition 2.2, \( c \) is not null homotopic.
Q.E.D.
4 the case $n$ and $m$ are odd

In this section we investigate the induced cohomology map of $\phi_{i,j}$ for $(i, j) \in (\mathbb{Z}/8\mathbb{Z})^2$, such that, $i + j = 2$.

We start from the next lemma.

**Lemma 4.1.** Assume $a \in H^u(\Omega^i BO)$ is primitive and $\phi_{i,j}^*(a) = \sum_{s+t=u} b_s \otimes c_t$ where $|b_s| = s$ and $|c_t| = t$. Then $b_s$ and $c_t$ are primitive.

**Proof.** Since for any $\alpha, \beta, \gamma \in \widetilde{KO}(X)$,

$$(p_1^*\alpha \oplus p_2^*(\beta)) \otimes p_3^*(\gamma) = (p_1^*\alpha \otimes p_3^*(\gamma)) \oplus (p_2^*(\beta) \otimes p_3^*(\gamma))$$

where $p_i : X \times X \times X \to X$ is the projection to $i$-th component, the next diagram commutes.

\[
\begin{array}{ccc}
\Omega^i BO \times \Omega^j BO \times \Omega^i BO & \xrightarrow{\Omega^i\mu \times 1} & \Omega^i BO \times \Omega^i BO \\
\downarrow(1 \times T \times 1) \circ (1 \times 1 \times \Delta) & & \downarrow (1 \times 1 \times \Delta) \\
\Omega^i BO \times \Omega^j BO \times \Omega^i BO \times \Omega^i BO & & \Omega^i BO \\
\downarrow \hat{\phi}_{i,j} \times \hat{\phi}_{i,j} & & \downarrow \hat{\phi}_{i,j} \\
\Omega^{i+j} BO \times \Omega^{i+j} BO & \xrightarrow{\Omega^{i+j}\mu} & BO \\
\end{array}
\]

Here $T$ is the transposition map, $\Delta$ is the diagonal map and $\mu : BO \times BO \to BO$ is the classifying map of $\eta_\infty \times \eta_\infty$ over $BO \times BO$. Further, $\hat{\phi}_{i,j}$ is the next composition:

$$(\Omega^i BO \times \Omega^j BO \to \Omega^i BO \wedge \Omega^j BO \to \Omega^{i+j} BO).$$

Let $a \in H^u(\Omega^i+j BO)$ be a primitive element. Then we have

\[
(1 \otimes \Delta^*) \circ (1 \otimes T^* \otimes 1) \circ (\hat{\phi}_{i,j} \otimes \hat{\phi}_{i,j}) \circ \mu^*(a) = (1 \otimes \Delta^*) \circ (1 \otimes T^* \otimes 1) \circ (\hat{\phi}_{i,j} \otimes \hat{\phi}_{i,j})(a \otimes 1 + 1 \otimes a) = (1 \otimes \Delta^*) \circ (1 \otimes T^* \otimes 1)(\sum b_s \otimes c_t \otimes 1 \otimes 1 + \sum 1 \otimes 1 \otimes b_s \otimes c_t) = (1 \otimes \Delta^*) (\sum b_s \otimes 1 \otimes c_t + \sum 1 \otimes b_s \otimes c_t) = (\sum (b_s \otimes 1 + 1 \otimes b_s) \otimes c_t).
\]
Also
\[(\mu^* \otimes 1) \circ \hat{\phi}_{i,j}^*(a) = (\mu^* \otimes 1)(\sum b_s \otimes c_t) = \sum \mu^*(b_s) \otimes c_t.\]

The above diagram says that these are the same. Therefore it occurs that \(\mu^*(b_s) = b_s \otimes 1 + 1 \otimes b_s\); that is, \(b_s\) is primitive. Similarly we can prove that \(c_t\) is primitive.

Q.E.D.

**Theorem 4.2.** Let \(i + j = 2\) and \(n\) and \(m\) be odd. Assume \(\phi_{i,j}(\alpha_{n+m-2}) = \sum b_s \otimes c_t\) where \(|b_s| = s\) and \(|c_t| = t\). If \(\binom{n+m-2}{n-1} \equiv 0 \mod 2\), then \(b_{n-1} \otimes c_{m-1} = 0\).

**Proof.** From assumption, \((i, j)\) is \((1, 1), (2, 0), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3)\) or \((0, 2)\). From the symmetricity, we shall look in to the cases \((i, j) = (1, 1), (2, 0), (3, 7), (4, 6)\) and \((5, 5)\).

For \(\phi_{3,7}, \phi_{5,5}\), the proof is easy. From the assumption, \(n - 1\) and \(m - 1\) are even and by Lemma 4.1, \(b_{n-1}\) and \(c_{m-1}\) are primitive. On the other hand, it is known that all of the non-zero primitive elements of \(\Omega^2 BO\), \(\Omega^5 BO\) are in odd degrees.[7] Thus \(b_{n-1} \otimes c_{m-1} = 0\).

To start the proof for \(\phi_{2,0}\), we investigate \(\phi_{0,0}^*\).

Let \(N = 2^r, r \in \mathbb{N}\) and \(\eta \in \widetilde{KO}(BO(N) \wedge BO(N))\) be the class of
\[\eta = (\xi_N - N) \hat{\otimes} (\xi_N - N)\]

We calculate the total Stiefel-Whitney class of \(\eta\) in \(H^*(B(\mathbb{Z}/2\mathbb{Z})^N \wedge B(\mathbb{Z}/2\mathbb{Z})^N) \supset H^*(BO \wedge BO)\). Let \(t_1, \ldots, t_N\) and \(t'_1, \ldots, t'_N\) be the generator of \(H^*(B(\mathbb{Z}/2\mathbb{Z})^N \wedge B(\mathbb{Z}/2\mathbb{Z})^N)\) where \(t_i\) corresponds to the first component and \(t'_i\) corresponds to the second. Then \(w_k = \sigma_k(t_1, \ldots, t_N)\) and \(w'_k = \sigma_k(t'_1, \ldots, t'_N)\) (\(1 \leq k \leq N\)) are the generators of \(H^*(BO \wedge BO)\) where \(\sigma_k\) is \(k\)-th fundamental symmetric polynomial. (We put \(w_0 = 1\)\).

Also we set \(S'_i = \sum_{i=1}^{N} t'_i\).

**Lemma 4.3.** The total Stiefel-Whitney class of \(\eta\) satisfies
\[w(\eta) = 1 + \sum_{k=0}^{N-1} \sum_{l=0}^{N-k} \binom{N-k}{l} w_{N-k} \otimes S'_l \mod \mod (w_1 \otimes 1, w_2 \otimes 1, \cdots, w_N \otimes 1)^2\]
in \(H^*(BO(N) \wedge BO(N))\) for \(\ast < N\).
Proof. Since  
\[ \eta = \xi_N \otimes \xi_N - \xi_N \otimes N - N \otimes \xi_N + N \otimes N, \]
we can see that  
\[ w(\eta) = \prod_{1 \leq i \leq N, 1 \leq j \leq N} (1 + t_i + t_j') \prod_{1 \leq i \leq N} (1 + t_i)^{-N} \prod_{1 \leq j \leq N} (1 + t_j')^{-N}. \]

Here in the part of degrees less than \( N \), \((1+t_i)^{-N} = (1+t_iN)^{-1} = 1 \) and also \((1+t'_j)^{-N} = 1 \). Therefore modulo \( \bigoplus_{i \geq N} H(i(B(\mathbb{Z}/2\mathbb{Z})^N \times B(\mathbb{Z}/2\mathbb{Z})^N)) \), we obtain that  
\[ w(\eta) = \prod_{1 \leq i \leq N, 1 \leq j \leq N} (t_i + 1 + t_j') \]
\[ \equiv \prod_{j=1}^{N} (\sum_{k=0}^{N} w_k (1 + t_j')^{N-k}) \]
\[ = \prod_{j=1}^{N} (1 + \sum_{k=0}^{N} \sum_{l=0}^{N-k} (N-k \choose l) w_k t_j'^l). \]

We proceed the calculation modulo \((w_1 \otimes 1, w_2 \otimes 1, \cdots, w_N \otimes 1)^2\) and obtain  
\[ w(\eta) \equiv 1 + \sum_{k=1}^{N} \sum_{l=1}^{N-k} (N-k \choose l) w_k S_l' \]
\[ \equiv 1 + \sum_{1 \leq k, 1 \leq l, k + l \leq N} \left( N-k \choose l \right) w_k S_l'. \]

Q.E.D.

**Lemma 4.4.** Let \( k, l, r \in \mathbb{N} \). If \( 2^r > k + l \), then \( \left( 2^r - k \right) \choose l \equiv \left( k + l - 1 \right) \mod 2 \).

**Proof.** We set the binary expansion of \( k - 1, l \) as  
\[ k - 1 = \sum_{0 \leq i \leq r-1} \epsilon_i 2^i, \quad l = \sum_{0 \leq i \leq r-1} \delta_i 2^i. \]

Then we have  
\[ \left( 2^r - k \right) \choose l \equiv \left( 2^r - 1 \right) \choose l \equiv \prod_{0 \leq i \leq r-1} \left( 1 - \epsilon_i \right). \]

Therefore \( \left( 2^r - k \right) \choose l \equiv 0 \) if and only if, for some \( i, \left( 1 - \epsilon_i \right) \equiv 0 \), i.e., \( \epsilon_i = \delta_i = 1 \).
Assume, for some \(i \ (0 \leq i \leq r - 1)\), \(\epsilon_i = \delta_i = 1\). Then let \(i_0\) be the smallest such a number. Then \(i_0\)-th coefficient of the binary expansion of \(k + l - 1\) is 0, while \(\delta_{i_0} = 1\). Thus we have \((k + l - 1) \equiv 0\).

Vice versa if, for any \(i \ (0 \leq i \leq r - 1)\), not both \(\epsilon_i\) and \(\delta_i\) are 1, then \((k + l - 1) \not\equiv 0\).

Therefore \((2^r - k) \equiv (k + l - 1) \mod 2\).

Q.E.D.

Since \(\phi_{0,0}\) is the classifying map of \(\eta_\infty \hat{\otimes} \eta_\infty\), Lemma 4.3 implies that

\[
\phi_{0,0}^*(w_i) = \sum_{k+l=i, \ k \text{ even}} \binom{2^r - k}{l} w_k \otimes S'_l
\]

where \(r\) is sufficiently large.

Therefore

\[
\phi_{0,0}^*(w_i) = \sum_{k+l=i, \ k \text{ even}} \binom{k+l-1}{l} \Sigma^2 a_{k-2} \otimes S'_l
\]

since

\[
\kappa_2^*(w_k) = \begin{cases} 
\Sigma^2 a_{k-2} & k \text{ : even} \\
0 & k \text{ : odd} 
\end{cases}
\]

and \(\kappa_2^*(\text{decomposable element}) = 0\).

From definition, \(\phi_{2,0} = \text{Ad}^2(\kappa_2 \wedge \text{Id}) \circ \phi_{0,0}\) and then we have

\[
\phi_{2,0}^*(a_{4i+2}) = \sum_{k+l=4i+2, \ k \text{ even}} \binom{k+l}{l} a_k \otimes S_l
\]

here we remark that \(\binom{k+l+1}{l} = \binom{k+l}{l}\) when \(k\) and \(l\) are even. From (6), and since \(a_{4k} = a_{2k}^2\), it occurs that

\[
\phi_{2,0}^*(a_{2p(4i+2)}) = \sum_{k+l=4i+2, \ k \text{ even}} \binom{k+l}{l} a_k^{2^p} \otimes S_l^{2^p},
\]

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Thus the coefficient of $b_{n-1} \otimes c_{m-1}$ in $\phi_{2,0}^* (a_{n+m-2})$ is 0 when $\binom{n+m-2}{n-1} = 0$ and the statement is true for $\phi_{2,0}$.

Second case is $\phi_{1,1}$. Consider the composition of following maps.

$$\Sigma \Omega BO \wedge \Sigma \Omega BO \xrightarrow{\kappa_1 \wedge \kappa_1} BO \wedge BO \xrightarrow{\phi_{0,0}} BO.$$  

From (5) and since $\kappaconst^*(\text{decomposable element}) = 0$ and

$$\kappaconst^*((w_k)) = \sum x_{k-1} \quad k: \text{odd}$$

the induced cohomology map of this composition can be obtained as

$$(\kappa_1 \wedge \kappa_1)^* \circ \phi_{0,0}^* (w_i) = (\kappa_1 \wedge \kappa_1)^* \left( \sum_{k+l=i} (k+l-1) \left( S_l \otimes w_k \right) \right)$$  \hspace{1cm} (7)

$$= \sum_{k+l=i, \text{odd}} (k+l-1) \sum x_{l-1} \otimes \sum x_{k-1}. \hspace{1cm} (8)$$

Here we remark that $\binom{k+l-1}{l} = 0$ when $l$ is odd and $k$ is even. Thus it occurs that

$$(\kappa_1 \wedge \kappa_1)^* \circ \phi_{0,0}^* (w_i) = \sum_{k+l=i, \text{odd}, \text{odd}} (k+l-1) \sum x_{l-1} \otimes \sum x_{k-1}. \hspace{1cm} (9)$$

Similarly as the case of $\phi_{2,0}$, $\phi_{1,1} = \text{Ad}^2 (\kappa_1 \wedge \kappa_1 \circ \phi_{0,0})$ and from (9) we have

$$\phi_{1,1}^* (\alpha_{2i+2}) = \sum_{k+l=4(i+1), \text{odd}, \text{odd}} (k+l) x_{l-1} \otimes x_{k-1}$$  \hspace{1cm} (10)

$$= \sum_{k+l=4i+2, \text{even}, \text{even}} (k+l) x_l \otimes x_k.$$  

And also

$$\phi_{1,1}^* (\alpha_{2p(4i+2)}) = \sum_{k+l=4i+2, \text{even}, \text{even}} (k+l) x_l^{2p} \otimes x_k^{2p}. \hspace{1cm} (11)$$

Thus the coefficient of $b_{n-1} \otimes c_{m-1}$ in $\phi_{1,1}^* (a_{n+m-2})$ is also 0 when $\binom{n+m-2}{n-1} = 0$ and the statement is true for $\phi_{1,1}$.
The final case is $\phi_{4,6}$. Let $\xi_n^R$, $\xi_n^C$ and $\xi_n^H$ be the universal bundle over $BO(n)$, $BU(n)$ and $BSp(n)$ respectively and put

$$\eta_n^R = \xi_n^R - n, \eta_n^C = \xi_n^C - n, \eta_n^H = \xi_n^H - n.$$ 

and

$$\eta_\infty^R = \lim_{n \to \infty} \xi_n^R - n, \eta_\infty^C = \lim_{n \to \infty} \xi_n^C - n, \eta_\infty^H = \lim_{n \to \infty} \xi_n^H - n.$$ 

Also set $c$ be the classifying map of $(\eta_\infty^R)_C$, complexification of $\eta_\infty^R$, $c'$ be the classifying map of $\eta_\infty^H$ as a complex vector bundle and $\psi$ be the classifying map of $\eta_\infty^C \hat{\otimes} \eta_\infty^C$ over $BU \wedge BU$.

We start from the next lemma.

**Lemma 4.5.** The next diagram commutes.

$$\begin{array}{ccc}
BSp \wedge BSp & \xrightarrow{c' \wedge c} & BU \wedge BU \\
\phi_{4,4} & \downarrow & \psi \\
BO & \xrightarrow{c} & BU
\end{array}$$

**Proof.** Consider the next composition:

$$\Sigma^4 BSp \wedge \Sigma^4 BSp \xrightarrow{K_4 \wedge K_4} BO \wedge BO \xrightarrow{\phi_{0,0}} BO \xrightarrow{c} BU.$$ 

Here in K-theory, $c^*(\eta_\infty^C) = (\eta_\infty^R)_C$ and $\phi_{0,0}^*((\eta_\infty^R)_C) = (\eta_\infty^R)_C \hat{\otimes} (\eta_\infty^R)_C$. Also it is known that $\kappa_4^*((\eta_\infty^R)_C) = (\zeta_H - \mathbb{H}) \hat{\otimes} (\zeta_H - \mathbb{H}) \hat{\otimes} \eta_\infty^H \hat{\otimes} \eta_\infty^H$. Therefore above composition pulls back $\eta_\infty^C$ to $(\zeta_H - \mathbb{H}) \hat{\otimes} (\zeta_H - \mathbb{H}) \hat{\otimes} (\zeta_H - \mathbb{H}) \hat{\otimes} \eta_\infty^H.$

On the other hand consider the next composition:

$$\Sigma^8 BSp \wedge BSp \xrightarrow{\Sigma^8(c' \wedge c)} \Sigma^8 BU \wedge BU \xrightarrow{\Sigma^8 \psi} \Sigma^8 BU \xrightarrow{\kappa_8^s} BU.$$ 

Here $\kappa_8^s$ is defined as follows. From Bott Periodicity, we know that $\Omega^2 BU \cong BU \times \mathbb{Z}$. Thus there exists a map $\kappa_2: \Sigma^2 BU \to BU$ which satisfies $\text{Ad}^2 \kappa_2$ is the inclusion map $BU \to \Omega^2 BU$. One can easily verify that

$$\kappa_2 \circ \Sigma^2 \kappa_2 \circ \cdots \Sigma^{2i-2} \kappa_2 \simeq \kappa_2$$

and it is known that in K-theory $\kappa_2^s(\eta_\infty^C) = (\zeta_C - \mathbb{C}) \hat{\otimes} \eta_\infty^C$ where $\zeta_C$ is the canonical line bundle over $\mathbb{C}P^1$. Therefore $\kappa_8^s = (\zeta_C - \mathbb{C})^4 \hat{\otimes} \eta_\infty^C$. Now we can see that the above composition pulls back $\eta_\infty^C$ to $(\zeta_C - \mathbb{C})^4 \hat{\otimes} \eta_\infty^H \hat{\otimes} \eta_\infty^H$. 

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Since $K^{-4}(pt) = \mathbb{Z}$ and the second Chern class of $-(\zeta_H - \mathbb{H})$ and $(\zeta_C - \mathbb{C})^2$ coincide, we see that the above two compositions are homotopic each other.

Take the $\text{Ad}^g$ of two compositions and we obtain

$$c \circ \phi_{4,4} \simeq \psi \circ c'$$

Q.E.D.

Refer to the diagram of Lemma 4.5. We want to calculate $\phi_{4,4}(w_i)$. As we have done in the proof of Lemma 4.3, let $N = 2^r$, $r \in \mathbb{N}$ and $\theta \in \widetilde{K}(BU(2N) \times BU(2N))$ be the class of $\theta = (\xi_{2N}^C - 2N) \otimes (\xi_{2N}^C - 2N)$ where $\xi_{2N}^C$ is the universal vector bundle over $BU(2N)$. Also let $\psi_N$ be the classifying map of $\theta$.

First, we calculate the total Chern class of $\theta$ in $H^*(BT^{2N} \times BT^{2N}) \supset H^*(BU(2N) \times BU(2N))$. Let $t_1, \ldots, t_{2N}, t'_1, \ldots, t'_{2N} \in H^*(BT^{2N} \times BT^{2N})$ be the generators as usual. Then in the part of degree less than $4N$,

$$\psi_N^*(1 + \sum_{i=1}^{\infty} c_i) = \prod_{1 \leq i \leq 2N, 1 \leq j \leq 2N} (1 + t_i + t'_j).$$

Now we proceed the calculations of $(c' \wedge c')^*\psi_N^*(1 + \sum_{i=1}^{\infty} c_i)$ in $H^*(BT^N \times BT^N) \supset H^*(BSp(N) \times BSp(N))$. Let $s_1, \ldots, s_N, s'_1, \ldots, s'_N \in H^*(BT^N \times BT^N)$ be the generators. Then we can see

$$(c' \wedge c')^*\psi_N^*(1 + \sum_{i=1}^{\infty} c_i) = (c' \wedge c')^* \left( \prod_{1 \leq i \leq 2N, 1 \leq j \leq 2N} (1 + t_i + t'_j) \right)$$

$$= \prod_{1 \leq i \leq N, 1 \leq j \leq N} (1 + s_i + s'_j)(1 + s_i - s'_j)(1 - s_i + s'_j)(1 - s_i - s'_j)$$

$$= \prod_{1 \leq i \leq N, 1 \leq j \leq N} (1 + s_i + s'_j)^4$$

$$= \left\{ \prod_{1 \leq i \leq N, 1 \leq j \leq N} (1 + s_i^2 + s'_j^2) \right\}^2.$$
than 4\(N\),

\[
(c' \wedge c')^* \psi_N^*(1 + \sum_{i=1}^{\infty} c_i) = \phi_{4,4}^* c^*(1 + \sum_{i=1}^{\infty} c_i)
\]

\[
= \phi_{4,4}^* (1 + \sum_{i=1}^{\infty} w_i^2)
\]

\[
= \phi_{4,4}^* (1 + \sum_{i=1}^{\infty} w_i)^2
\]

Since \(H^*(BSp \wedge BSp)\) is a subalgebra of a polynomial algebra, the square of any element in \(H^*(BSp \wedge BSp)\) does not vanishes. Therefore

\[
\phi_{4,4}^*(1 + \sum_{i=1}^{\infty} w_i) = \prod_{1 \leq i \leq N, 1 \leq j \leq N} (1 + s_i^2 + s_j^2)
\]

in the part of degree less than 2\(N\).

We set \(q'_k = \sigma_k(s'_{1}^2, \ldots, s'_{N}^2)\) (1 \( \leq k \leq N\)) which are the generators of \(H^*(BSp(N))\) and \(Q_l = \sum_{i=1}^{N} s_i \overline{2}^l\) which is the primitive element of \(H^*(BSp(N))\). Now we have in the part of degrees less than 2\(N\)

\[
\phi_{4,4}^*(1 + \sum_{i=1}^{\infty} w_i) = \prod_{1 \leq i \leq N, 1 \leq j \leq N} (1 + s_i^2 + s_j^2)
\]

\[
= \prod_{i=1}^{N} \left( \sum_{k=0}^{N} q_{N-k}^l \right)
\]

\[
= \prod_{i=1}^{N} \left( 1 + \sum_{k=0}^{N-1} \sum_{l=0}^{k} \binom{k}{l} s_i \overline{2}^l q_{N-k}^l \right)
\]

Now we proceed the calculations modulo \((q'_1, \ldots, q'_N)^2\).

\[
\phi_{4,4}^*(1 + \sum_{i=1}^{\infty} w_i) \equiv 1 + \sum_{k=0}^{N-1} \sum_{l=1}^{k} \binom{k}{l} Q_l q_{N-k}^l
\]

\[
\equiv 1 + \sum_{k=1}^{N} \sum_{l=1}^{N-k} \binom{N-k}{l} Q_l q_k'
\]

\[
\equiv 1 + \sum_{i=1}^{N} \sum_{1 \leq k, 1 \leq l, k + l = i} \binom{N-k}{l} Q_l q_k'
\]
This leads us to the next lemma.

**Lemma 4.6.** Modulo \((1 \otimes q_1, 1 \otimes q_2, 1 \otimes q_3, \ldots)^2\),

\[
\phi_{4,4}^*(w_i) = \begin{cases} 
\sum_{1 \leq k, 1 \leq l, k + l = j} (k + l - 1) Q_l \otimes q_k & i = 4j \\
0 & i \not\equiv 0 \pmod{4}
\end{cases}
\]

where \(H^*(BSp) = \mathbb{Z}/2\mathbb{Z}[q_1, q_2, q_3, \ldots]\) and \(Q_l \in H^*(BSp)\) is the primitive element of degree \(4l\).

Let \(\kappa' : \Sigma^2\Omega^6 BO \to \Omega^4 BO\) be the map which satisfies \(\text{Ad}^2(\kappa') = \text{Id}_{\Omega^2 BO}\). Then it can be easily verified that \(\text{Ad}^2(\phi_{4,4} \circ \text{Id}_{\Omega^2 BO} \land \kappa') = \phi_{4,6}\). Since

\[
\kappa'^*(q_l) = \Sigma^2 b_{4l-2},
\]

where \(H^*(\Omega^2 BSp) = \bigwedge((b_2, b_4, b_6, \ldots)\) and \(b_{4l-2}\) is primitive, it occurs that

\[
(\text{Id}_{\Omega^2 BO} \land \kappa')^* \phi_{4,4}^*(w_{4i}) = \sum_{1 \leq k, 1 \leq l, k + l = i} (k + l - 1) Q_l \otimes \Sigma^2 b_{4k-2}
\]

and

\[
\phi_{4,6}^*(a_{4i-2}) = \sum_{1 \leq k, 1 \leq l, k + l = j} (k + l - 1) Q_l \otimes b_{4k-2}.
\]

Remark that \((k + l - 1) = \left(\frac{4k + 4l - 4}{4l}\right) = \left(\frac{4k + 4l - 2}{4l}\right)\) and

\[
\phi_{4,6}^*(a_{2p(4i-2)}) = \sum_{1 \leq k, 1 \leq l, k + l = j} \left(\frac{4k + 4l - 2}{4l}\right) Q_l^{2p} \otimes b_{4k-2}^{2p}.
\]

Therefore the statement is also true for \(\phi_{4,6}\).

Q.E.D. (Theorem 4.2)

From Theorem 3.3 and Theorem 4.2, the next theorem follows.

**Theorem 4.7.** Assume neither \(n - 1\) nor \(m - 1\) is a power of 2 and both \(n\) and \(m\) are odd. If \(\binom{n + m - 2}{n - 1} \equiv 0 \pmod{2}\), \((n, m)\) is Spin-regular.
5 the case $n$ and $m$ are even

In this section we use integral cohomology. Consider the next diagram.

\[
\begin{array}{ccc}
S^{n-1} & \xrightarrow{i_n} & Spin(n) & \xrightarrow{\pi_n} & S^{n-1} \\
\downarrow{p_n'} & & \downarrow{p_n} & & \cong \\
RP^{n-1} & \xrightarrow{i_n} & SO(n) & \xrightarrow{\pi_n'} & S^{n-1}
\end{array}
\]

Here $\pi_n$, $\pi_n'$ is the map obtained from $Spin(n) \to Spin(n)/Spin(n-1) = S^{n-1}$ and $SO(n) \to SO(n)/SO(n-1) = S^{n-1}$ respectively. Also $i_n$ is the inclusion map defined as follows. Let $l \in RP^{n-1}$ be a line and let $e \in l$ be a unit vector. Then $i_n(l) = i_n'(l_0)i_n'(l)$ where $i_n'(l)(v) = v - 2(v,e)e$ and $l_0$ is the base point of $RP^{n-1}$. We set $p_n' : S^{n-1} \to RP^{n-1}$ be the usual covering map then there is a map $\tilde{i}_n$ which makes diagram commutative. Moreover, when $n = 4$, $\pi_n$ has a section $\epsilon : S^{n-1} \to Spin(n)$, that is, $\pi_n \circ \epsilon = Id$.

We set $c_{n-1}$ as the generator of $H^*(S^{n-1};\mathbb{Z})$ and take $\delta \in H^*(Spin(n) \wedge Spin(m);\mathbb{Z})$ as $\delta = (\pi_n \wedge \pi_m)^*(c_{n-1} \otimes c_{m-1})$.

**Lemma 5.1.** If $n$ and $m$ are even and neither $n$ nor $m$ is 4,

\[H^{n+m-2}(Spin(n) \wedge Spin(m);\mathbb{Z}) = \langle \delta \rangle \oplus \text{Ker}(\tilde{i}_n \wedge \tilde{i}_m)^*.\]

**Proof.** Since $n$ is even, $i_n^*\pi_n^*(c_{n-1})$ is the generator of $H^{n-1}(RP^{n-1};\mathbb{Z}) \cong \mathbb{Z}$. Therefore

\[\tilde{i}_n^*\pi_n^*(c_{n-1}) = p_n'^*i_n^*\pi_n^*(c_{n-1}) = 2c_{n-1},\]  

that is, $\tilde{i}_n \wedge \tilde{i}_m^*(\delta) = 4c_{n-1} \otimes c_{m-1}$.

Because $p_n'^* : H^{n-1}(RP^{n-1};\mathbb{Z}/2\mathbb{Z}) \to H^{n-1}(S^{n-1};\mathbb{Z}/2\mathbb{Z})$ is a 0-map and $\tilde{i}_n^* \circ p_n^* = p_n'^* \circ i_n^*$, we have $\tilde{i}_n^* \circ p_n^* = 0$ in mod 2 cohomology. Further, since, when $n \neq 4$, $p_n^* : H^{n-1}(SO(n);\mathbb{Z}/2\mathbb{Z}) \to H^{n-1}(Spin(n);\mathbb{Z}/2\mathbb{Z})$ is epic, this implies that $\tilde{i}_n^* : H^{n-1}(Spin(n);\mathbb{Z}/2\mathbb{Z}) \to H^{n-1}(S^{n-1};\mathbb{Z}/2\mathbb{Z})$ is also a 0-map. Therefore $\text{Im}(\tilde{i}_n^*) \subset \langle 2c_{n-1} \rangle$ in integral cohomology.

Now we obtain that $\text{Im}(\tilde{i}_n \wedge \tilde{i}_m)^* = \langle 4c_{n-1} \otimes c_{m-1} \rangle = \langle (\tilde{i}_n^* \wedge \tilde{i}_m^*)(\delta) \rangle$ and from the freeness of $H^{n+m-2}(S^{n+m-2};\mathbb{Z})$ the statement follows.

Q.E.D.

**Lemma 5.2.** If $n = 4$ and $m$ are even and $m \neq 4$,

\[H^{n+m-2}(Spin(n) \wedge Spin(m);\mathbb{Z}) = \langle \delta \rangle \oplus \text{Ker}(\epsilon \wedge \tilde{i}_m)^*.\]
Proof. From (12) and $\epsilon^*\pi_4^*(c_3) = c_3$, 

$$(\epsilon \wedge \tilde{i}_m)^*(\delta) = 2c_{n-1} \otimes c_{m-1}.\)$$

As seen in the proof of previous lemma, Im$\tilde{i}_m^* \subset \langle 2c_{m-1} \rangle$ in integral cohomology and since $\epsilon$ is a section, Im$\epsilon^* = \langle c_3 \rangle$.

Now it follows that Im$(\epsilon \wedge \tilde{i}_m)^* = \langle 2c_3 \otimes c_{m-1} \rangle = \langle (\epsilon \wedge \tilde{i}_m)^*(\delta) \rangle$ and from the freeness of $H^{n+m-2}(S^{n+m-2}; \mathbb{Z})$ the statement follows.

Q.E.D.

Theorem 5.3. Assume neither $n - 1$ nor $m - 1$ is a power of 2, both $n$ and $m$ are even, $n + m \equiv 0 \mod 4$ and $n + m \geq 16$. Then $(n, m)$ is Spin-regular.

Proof. We use Proposition 2.2. Let $x: Spin(n) \vee Spin(m) \to \Omega Spin$ satisfies $x^*(\alpha_{n+m-2}) = x_{n-1} \otimes x_{m-1}$ in mod 2 cohomology. Then there exists $\eta \in \tilde{KO}(\Sigma^2 Spin(n) \vee Spin(m))$ which satisfies

$$w_{n+m}(\eta) = \Sigma^2 x_{n-1} \otimes x_{m-1}. \quad (13)$$

Here, since Pontrjagin square acts trivially in $H^*(\Sigma^2 Spin(n) \vee Spin(m); \mathbb{Z})$, by the second formula of Wu [12],

$$\rho_4(P_{n+m}(\eta)) = w'_{n+m}(\eta), \quad (14)$$

where $w'_{n+m}$ is the image of $w_{n+m}$ under the coefficient monomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ and $\rho_4$ is the map of mod 4 reduction.

When neither $n$ nor $m$ is 4, from (13), (14) and Lemma 5.1, we can see that

$$P_{n+m}(\eta) = \Sigma^2((4k + 2)\delta + \alpha),$$

where $\alpha \in \text{Ker}(i_n \wedge i_m)^*$ and we obtain

$$P_{n+m}(\Sigma^2(i_n \wedge i_m)^*(\eta)) = (16k + 8)c_{n+m}.$$

When $n = 4$ and $m \neq 4$, (13), (14) and Lemma 5.2 imply that

$$P_{n+m}(\eta) = \Sigma^2((4k + 2)\delta + \beta),$$

where $\beta \in \text{Ker}(\epsilon \wedge \tilde{i}_m)^*$ and we have

$$P_{n+m}(\Sigma^2(\epsilon \wedge \tilde{i}_m)^*(\eta)) = (8k + 4)c_{n+m}.$$

But for the generator $\eta_0$ of $\widetilde{KO}(S^{n+m})$, $P_{n+m}(\eta_0)$ is divisible by $(n+m)/2 - 1)!$. [1] When $n + m \geq 16$ this is a contradiction and the statement follows.
Theorem 5.4. Assume neither $n-1$ nor $m-1$ is a power of 2, both $n$ and $m$ are even. If $n + m = 12$ or $n + m \equiv 2 \mod 4$. Then $(n, m)$ is Spin-regular.

Proof. We use Proposition 2.2. Let $x : \text{Spin}(n) \wedge \text{Spin}(m) \to \Omega \text{Spin}$ be the arbitrary continuous map.

When $n + m \equiv 2 \mod 4$, that is, $n+m-2$ is divisible by 4, $x^*(\alpha_{n+m-2}) = x^*(\alpha_{n+m-2})^2$ in mod 2 cohomology. Thus $x^*(\alpha_{n+m-2})$ can be written in the form $\sum \alpha \otimes \beta$ where $\alpha$ and $\beta$ are decomposable. Therefore $x^*(\alpha_{n+m-2}) \neq x_{n-1} \otimes x_{m-1}$.

Now let $n + m = 12$ and $n \leq m$. When $n \neq 4$, $x^*(\alpha_6) = x_3 \otimes x_3$ or 0 and when $n = 4$, $x^*(\alpha_6) = z \otimes x_3$, $x_3 \otimes x_3$ or 0. We can see

$$\text{Sq}^2 x^*(\alpha_6) = x^*(\text{Sq}^2 \alpha_6) = x^*(\alpha_8) = x^*(\alpha_2)^4 = 0$$

while

$$\text{Sq}^2 x_3 \otimes x_3 = x_5 \otimes x_3 + x_3 \otimes x_5,$$

$$\text{Sq}^2 z \otimes x_3 = z \otimes x_5.$$

So $x^*(\alpha_6) = 0$ and we have

$$x^*(\alpha_{10}) = x^*(\text{Sq}^4 \alpha_6) = \text{Sq}^4 x^*(\alpha_6) = 0.$$

Q.E.D.

From Proposition 2.2, Theorems 4.7, 5.3, 5.4, we finally obtain Theorem 1.3.

6 (3, 4k + 1) is Spin-irregular

In this section we shall give the proof of Theorem 1.4 which requires that $(3, 4k+1)$ is Spin-irregular.

Since there are embeddings $\text{Spin}(3) \to \text{Spin}(4k + 3)$, $\text{Spin}(4k + 1) \to \text{Spin}(4k + 3)$ where any element of $\text{Spin}(3)$ and any element of $\text{Spin}(4k) \subset \text{Spin}(4k + 1)$ exactly commute in $\text{Spin}(4k + 3)$. Let $A \in \text{Spin}(3)$, $B \in \text{Spin}(4k + 1)$, $C \in \text{Spin}(4k) \subset \text{Spin}(4k + 1)$. Then $A(BC)A^{-1}(BC)^{-1} = ABCA^{-1}C^{-1}B^{-1} = ABA^{-1}B^{-1}$ and the commutator of $A$ and $B$ is invariant under the right translation of $\text{Spin}(4k)$ on $B$.

Therefore there exists a map $\phi : \text{Spin}(3) \wedge (\text{Spin}(4k + 1)/\text{Spin}(4k)) \to \text{Spin}(4k + 3)$ such that $\phi \circ (1 \wedge \pi_{4k+1}) \simeq c$. See the diagram below. Remark that $\text{Spin}(3) \simeq S^3$ and $\text{Spin}(4k + 1)/\text{Spin}(4k) \simeq S^{4k}$. 

Q.E.D.
In the above diagram $\Omega SO/\text{SO}(4k + 3) \to \text{Spin}(4k + 3) \to \text{Spin}$ is a fibration and $i \circ e'$ is null homotopic. So there exists a map $\lambda : S^{4k+3} \to \Omega SO/\text{SO}(4k + 3)$, such that $\delta \circ \lambda \simeq e'$.

Since $\pi_{4k+4}(SO/\text{SO}(4k + 3)) \cong 0$ ([10]), $\pi_{4k+3}(\Omega SO/\text{SO}(4k + 3)) \cong 0$ and $\lambda$ is null homotopic.

Thus $c \simeq \delta \circ \lambda \circ (1 \wedge \pi_{4n+1}) \simeq *$ and Theorem 1.4 is proved.

References


