NON CONCENTRATION OF CURVATURE
NEAR SINGULAR POINTS
OF TWO VARIABLE ANALYTIC FUNCTIONS

SATOSHI KOIKE, TZEE-CHAR KUO AND LAURENTIU PAUNESCU

Abstract. In this paper we study the phenomenon of non concentration of curvature of level curves of two variable complex analytic function germs, and we characterise it in terms of tree models and topological types. In addition, we also discuss the relationship in the real case, between the phenomenon of non concentration of curvature and real tree models (or blow-analytic types). In particular, we give an example to demonstrate that the corresponding characterisation does not hold in the real case.

1. Introduction

Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be an analytic function germ not identically zero, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The singular point set of $f$ is contained in the zero-set $f^{-1}(0)$. What kind of relationship can we find between the singularity type of $f : \mathbb{K}^2 \to \mathbb{K}$ and its level curves $f = c$, $0 < |c| < \epsilon$, $\epsilon$ small? In this respect N. A'Campo made a profound observation. R. Langevin explored the A'Campo phenomenon and found (in [16]) an interesting relationship between the integration of the total Gaussian curvatures of the level curves of a complex analytic function and its Milnor number. In addition, E. Garcia Barroso and B. Teissier analysed the concentration of curvature of the level curves of a complex analytic function in [1], and J.-J. Risler investigated the curvature problem for the real Milnor fibre in [18]. In a couple of previous papers ([9], [10]), using the language of infinitesimals as introduced in [14],[15], we studied the A’Campo’s curvature bumps for both real and complex analytic functions.

The curvature formula in the real case is

\[ K_f^\mathbb{R}(x, y) := \pm \frac{\Delta_f(x, y)}{(f_x(x, y)^2 + f_y(x, y)^2)^{\frac{3}{2}}}, \]

and that in the complex case is

\[ K_f^\mathbb{C}(z, w) := -\frac{2|\Delta_f(z, w)|^2}{(|f_z(z, w)|^2 + |f_w(z, w)|^2)^3}, \]

where

\[ \Delta_f := 2f_xf_yf_{xy} - f_x^2f_{yy} - f_y^2f_{xx}. \]

These are known formulae for computing the Gaussian curvature of level curves $f = c$, $0 < |c| < \epsilon$ (cf. John A. Thorpe [19]).

Date: 4 March 2013.
1991 Mathematics Subject Classification. Primary 14B05, 14H50; Secondary 32S15, 58K20.
Key words and phrases. concentration of curvature, tree model, blow-analyticity, curvature tableland.
Let us consider the polynomial functions \( f_1, f_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) defined by
\[
    f_1(x, y) = x^2 - y^3, \quad f_2(x, y) = x^4 - y^5.
\]

Using the real formula above, we can easily see that if \(|c|\) is sufficiently small, the level curves of \( f_1 \) and \( f_2 \) are very close to be vertical in a (wide) horn-neighbourhood of the \( x \)-axis. Similarly the level curves of \( f_2 \) are also nearly horizontal in a horn-neighbourhood of the \( y \)-axis. Therefore the union of the level curves \( f_2 = \pm c, c \neq 0 \), looks like a rectangle outside some thin horn-like region tangent to the \( y \)-axis. Intuitively the concentration of curvature of \( f_2 \) happens in this thin region. On the other hand, we can see that the concentration of curvature of \( f_1 \) happens in a thin horn-neighbourhood of the \( y \)-axis. These phenomena are illustrated in the pictures below. (See also [9].)

In [6] the first named author and A. Parusiński gave a complete blow-analytic classification of two variable real analytic function germs in terms of their real tree model. See §3 for the definitions of blow-analytic equivalence and real tree model. The real tree models of the above \( f_1 \) and \( f_2 \) are drawn as follows:

\[
\begin{array}{cc}
\begin{array}{ccc}
+ & \frac{3}{2} & + \\
2 & 2 & \frac{3}{2} \\
(0, -1) & (0, 1) & \\
\end{array} & \quad & \begin{array}{ccc}
+ & \frac{5}{4} & + \\
4 & 4 & \frac{5}{4} \\
(0, -1) & (0, 1) & \\
\end{array}
\end{array}
\]

\[\mathbb{R}T(f_1) \quad \mathbb{R}T(f_2)\]

By an easy computation of the curvature using the above formula, we can observe that the concentration of curvature of \( f_1 \) and \( f_2 \) happens on the bars of height \( \frac{3}{2} \) and \( \frac{5}{4} \), respectively. Note that, in both cases, the concentration of curvature does not happen directly on the ground bar.

Remark 1.1. In [9] we made a more detailed analysis of the curvature of the level curves of the above \( f_1 \) and \( f_2 \). We let the trunks supporting the bars of height \( \frac{3}{2} \) in \( \mathbb{R}T(f_1) \) and of
height $\frac{5}{4}$ in $RT(f_2)$ grow upward and put provisional bars of height 2 and $\frac{3}{2}$ on the grown trunks in $RT(f_1)$ and $RT(f_2)$, respectively. In each case the concentration of curvature happens on the provisional bar. We do not elaborate on this observation in this paper.

In the real case the zero set $f^{-1}(0)$ can be just $\{(0, 0)\}$ as a set germ at the origin. Let us consider such polynomial functions $f_3, f_4 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ defined by

$$f_3(x, y) = x^4 + y^4, \quad f_4(x, y) = x^4 + y^6.$$ 

The real tree models of $f_3$ and $f_4$ are drawn below.

- $f_3(\mathbb{R}^2)$
- $f_4(\mathbb{R}^2)$

In the case of $f_3$ we can see that $|Kf_3(x, y)|$ on each level curve takes a maximum along some curves near $y = \pm x$, but non concentration of curvature happens. Note that there is no bar of height bigger than 1 in $RT(f_3)$. On the other hand, a level curve $f_4 = c, c > 0$, looks like a rectangle outside some thin horn-like region tangent to the $y$-axis. The concentration of curvature of $f_4$ happens in this thin region. We can see that the concentration of curvature of $f_4$ happens on the bars of height $\frac{3}{2}$.

The above observations give rise to natural questions.

**Question 1.** For a two variable real analytic function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is there any relationship between its real tree model and the concentration of curvature?

**Question 2.** For a two variable complex analytic function germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is there any relationship between its tree model and the concentration of curvature?

In this paper we give an affirmative answer to Question 2. Namely, we show that non concentration of curvature happens if and only if no bar of height bigger than 1 appears in the tree model $T(f)$ (Theorem 5.4). This condition is also equivalent to the homogeneous-likeness. (See §3 for the definition of homogeneous-like.) From this result, we can see that the appearance of concentration of curvature is a topological invariant in the complex case (Corollary 5.5).

On the other hand, we have a negative answer to Question 1. More precisely, the corresponding condition on the real tree model, which is equivalent to the homogeneous-likeness, implies non concentration of curvature, but the converse is not valid. Indeed, we give an example to demonstrate that non concentration of curvature does not always imply homogeneous-likeness (Proposition 5.7).

In order to introduce our notions of concentration of curvature and non concentration of curvature, we need to also consider the curvature of the level curve $f = 0$ in a punctured neighbourhood of $0 \in \mathbb{K}^2$, even if the zero-set is a singular locus of $f$. The sign of the
curvature is not essential for these notions. In the next section we define the non-directed curvature for the level curves \( f = c, \ |c| < \epsilon, \) in a punctured neighbourhood of \( 0 \in \mathbb{R}^2, \) and using this definition, we introduce the notions of concentration of curvature and non concentration of curvature in \( \S 5. \) In \( \S 3 \) we recall the notions of tree model and real tree model, mention some related results, and give a characterisation of the homogeneous-likeness in terms of tree models. In \( \S 4 \) we review several results on A’Campo curvature bumps proved in \([9, 10],\) which are necessary for the proofs of our main result in the complex case mentioned in \( \S 5. \)

Throughout our paper we will use the following convention and notations. We call an analytic function germ, including the zero locus. We first consider the real case.

Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) be an irreducible analytic function germ, and let \( f = g^m \) for \( m \geq 1. \) Then we have

\[
\Delta_f = m^3 g^{3(m-1)} \Delta_g \quad \text{and} \quad (f_x^2 + f_y^2)^{\frac{3}{2}} = \pm m^3 g^{3(m-1)} (g_x^2 + g_y^2)^{\frac{3}{2}}.
\]

Therefore, after cancellation of \( |g|^{3(m-1)} \), we have

\[
|K_f| = \frac{|\Delta_f|}{(f_x^2 + f_y^2)^{\frac{3}{2}}} = \frac{|\Delta_g|}{(g_x^2 + g_y^2)^{\frac{3}{2}}} = |K_g|
\]

in a punctured neighbourhood of \( 0 \in \mathbb{R}^2. \) Note that in case \( m \) is odd, \( K_f = K_g \) after cancellation of \( g^{3(m-1)}. \)

Let \( f(x, y) = x^2 y \) and \( g(x, y) = xy, \) then \( |K_f| \) does not coincide with \( |K_g| \) in a punctured neighbourhood of \( 0 \in \mathbb{R}^2. \) Nevertheless we still have some reasonable results on comparing their curvature. We first prepare some lemmas.

**Lemma 2.1.** Let \( g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) be an irreducible analytic function such that \( g^{-1}(0) \neq \{0\} \) as germs at \( 0 \in \mathbb{R}^2. \) Then \( g_x^2 + g_y^2 \) is not divisible by \( g. \)

**Proof.** Since \( g \) is irreducible, \( g \) has an isolated singularity at \( 0 \in \mathbb{R}^2 \) (cf. [11]). Suppose that \( g_x^2 + g_y^2 \) is divisible by \( g. \) Then \( g_x = 0 \) and \( g_y = 0 \) along \( g^{-1}(0). \) This contradicts the isolated singularity of \( g. \) \( \square \)
Remark 2.2. We cannot drop the assumption that \( g^{-1}(0) \neq \{0\} \) at \( 0 \in \mathbb{R}^2 \). Let \( g(x, y) = x^2 + y^2 \), \( g \) is irreducible and \( g^{-1}(0) = \{0\} \). On the other hand, \( g_x^2 + g_y^2 = 4(x^2 + y^2) \) is divisible by \( g \).

Let \( f, g, h : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) be analytic function germs such that \( f = g^m h \) for \( m \geq 2 \).

Suppose that \( g \) is an irreducible analytic function such that \( g^{-1}(0) \neq \{0\} \) as germs at \( 0 \in \mathbb{R}^2 \) and that \( h \) is not divisible by \( g \). Then we have the following lemmas.

**Lemma 2.3.** \( |\Delta_f| \) and \((f_x^2 + f_y^2)^{\frac{3}{2}}\) are divisible by \(|g^{3(m-1)}|\).

**Proof.** By an easy computation, we have

\[
\begin{align*}
  f_x &= (mg_x h + gh_x)g^{m-1}, \\
  f_y &= (mg_y h + gh_y)g^{m-1}, \\
  f_{xx} &= (mg_{xx} h + m(m-1)g_x^2 h + 2mg_x gh_x + g^2 h_{xx})g^{m-2}, \\
  f_{xy} &= (mg_{xy} h + m(m-1)g_x g_y h + mg_x gh_y + mg_y gh_x + g^2 h_{xy})g^{m-2}, \\
  f_{yy} &= (mg_{yy} h + m(m-1)g_y^2 h + 2mg_y gh_y + g^2 h_{yy})g^{m-2}.
\end{align*}
\]

Then we have

\[
\begin{align*}
  \Delta_f &= \{m^3 h^3 (2g_x g_y g_{xy} - g_x^2 g_{yy} - g_y^2 g_{xx}) + Hg\}g^{3(m-1)}, \\
  f_x^2 + f_y^2 &= \{m^2 h^2 (g_x^2 + g_y^2) + 2mh(g_x h_x + g_y h_y) + (h_x^2 + h_y^2)g^2\}g^{2(m-1)},
\end{align*}
\]

where \( H : \mathbb{R}^2 \to \mathbb{R} \) is an analytic function germ at \( 0 \in \mathbb{R}^2 \). Therefore \( |\Delta_f| \) and \((f_x^2 + f_y^2)^{\frac{3}{2}}\) are divisible by \(|g^{3(m-1)}|\).

After this, we put \( \tilde{\Delta}_f := \frac{|\Delta_f|}{|g|^{3(m-1)}} \) and \( f_\nabla := \frac{(f_x^2 + f_y^2)^{\frac{3}{2}}}{|g|^{3(m-1)}} \).

**Lemma 2.4.**

1. \( f_\nabla \neq 0 \) over \( g^{-1}(0) \setminus \{0\} \).

2. Let \( \tilde{K}_f := \frac{\tilde{\Delta}_f}{f_\nabla} \). Then \( \tilde{K}_f = |K_g| \) over \( g^{-1}(0) \setminus \{0\} \).

**Proof.**

1. By Lemmas 2.1 and 2.3, \( m^2 h^2 (g_x^2 + g_y^2) \neq 0 \) over \( g^{-1}(0) \setminus \{0\} \). Therefore \( f_\nabla \neq 0 \) over \( g^{-1}(0) \setminus \{0\} \).

2. Over \( g^{-1}(0) \setminus \{0\} \),

\[
\tilde{K}_f = \frac{|m^3 h^3 (2g_x g_y g_{xy} - g_x^2 g_{yy} - g_y^2 g_{xx})|}{(m^2 h^2 (g_x^2 + g_y^2))^{\frac{3}{2}}} = \frac{|2g_x g_y g_{xy} - g_x^2 g_{yy} - g_y^2 g_{xx}|}{(g_x^2 + g_y^2)^{\frac{3}{2}}} = |K_g|.
\]

**Remark 2.5.** Lemma 2.4 holds also for \( m = 1 \). In particular, (2) becomes the following:

\( K_f = \pm K_g \) over \( g^{-1}(0) \setminus \{0\} \), where the sign \( \pm \) depends on the sign of \( h \) at \((x, y) \in g^{-1}(0) \setminus \{0\}\).

Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) be an analytic function germ. Then \( f \) has a decomposition of the following form:

\[
(2.1) \quad f = f_1^{m_1} \cdots f_k^{m_k} h, \quad m_i \geq 1 \ (1 \leq i \leq k), \quad k \in \mathbb{N} \cup \{0\}, \quad f_i \neq f_j \ (i \neq j),
\]
where each \(f_i : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0), 1 \leq i \leq k\), is an irreducible analytic component of \(f\) such that \(f_i^{-1}(0) \neq \{0\}\) as germs at \(0 \in \mathbb{R}^2\), and \(h : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)\) is an analytic function germ such that \(h^{-1}(0) \subseteq \{0\}\) as germs at \(0 \in \mathbb{R}^2\). Note that \(k \in \mathbb{N}\) in case \(h\) is a unit.

Let us define the non-directed curvature of level curves of \(f\) as follows:

\[
\overline{K}_f(x, y) := \begin{cases} 
|K_f^2(x, y)| & \text{if } (x, y) \in \mathbb{R}^2 \setminus f^{-1}(0) \\
|K_f(x, y)| & \text{if } (x, y) \in f_j^{-1}(0) \setminus \{0\} \quad (1 \leq j \leq k).
\end{cases}
\]

The next proposition follows from Lemmas 2.3 and 2.4.

**Proposition 2.6.** \(\overline{K}_f\) is continuous in a punctured neighbourhood of \(0 \in \mathbb{R}^2\).

As mentioned in the introduction, using the concept of non-directed curvature, we shall define the notions of concentration of curvature and non concentration of curvature for two variable real analytic function germs in §5.

Concerning the reduction of the problem of concentration of curvature, it may be natural to ask the following question.

**Question 3.** Let \(f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)\) be an analytic function germ with a decomposition of form (2.1), and let \(g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)\) be the function germ defined by \(g = f_1 \cdots f_k h\). One can ask whether there are positive numbers \(0 < C_1 < C_2\) such that

\[
C_1|K_f| \leq |K_g| \leq C_2|K_f|.
\]

If the answer would be affirmative, then it would be enough to consider only reduced analytic function germs. Unfortunately this does not hold in general. In fact, we have

**Example 2.7.** Let \(f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)\) be real analytic function germs defined by

\[
f(x, y) = x^2(x - y^2), \quad g(x, y) = x(x - y^2).
\]

By simple computations, we can see

\[
|K_f| = \frac{|\Delta_f|}{(f_2^2 + f_y^2)^{\frac{3}{2}}} \approx \frac{1}{|y|^3}, \quad |K_g| = \frac{|\Delta_g|}{(g_x^2 + g_y^2)^{\frac{3}{2}}} \approx 1
\]

on the curve \(\{x = \frac{2}{3}y^2\}\). Therefore there does not exist \(C_1 > 0\) such that \(C_1|K_f| \leq |K_g|\).

We next consider the complex case.

**Lemma 2.8.** Let \(g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\) be an irreducible analytic function germ. Then \(g\) has an isolated singularity at \(0 \in \mathbb{C}^2\).

Let \(f, g, h : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\) be analytic function germs such that \(f = g^m h\) for \(m \geq 2\). Suppose that \(g\) is irreducible and \(h\) is not divisible by \(g\). Then we have the following.

**Lemma 2.9.** \(|\Delta_f|^2\) and \((|f_x|^2 + |f_w|^2)^3\) are divisible by \(|g|^{6(m-1)}\).

**Proof.** Similarly to Lemma 2.3, we have

\[
\Delta_f = \{m^3h^3(2g_xg_yg_{xy} - g^2_xg_{yy} - g^2_yg_{xx}) + Hg\}g^{3(m-1)},
\]

\[
|f_x|^2 + |f_w|^2 = (|mg_xh + gh_x|^2 + |mg_yh + gh_y|^2)|g|^{2(m-1)},
\]

where \(H : \mathbb{C}^2 \to \mathbb{C}\) is an analytic function germ at \(0 \in \mathbb{C}^2\). It follows that \(|\Delta_f|^2\) and \((|f_x|^2 + |f_w|^2)^3\) are divisible by \(|g|^{6(m-1)}\). \(\square\)
After this, we put \( \hat{\Delta}_f^C := -\frac{2|\Delta f|^2}{|g|^2(m-1)} \) and \( f_0^C := \frac{|f_0|^2+|f_0|^2}{|g|^2(m-1)} \). Using the same arguments as in Lemma 2.4, we can show a similar lemma.

**Lemma 2.10.** (1) \( f_0^C \neq 0 \) over \( g^{-1}(0) \setminus \{0\} \).
(2) Let \( \hat{K}_f^C := \frac{\Delta f}{f_0} \). Then \( \hat{K}_f^C = K_g^C \) over \( g^{-1}(0) \setminus \{0\} \).

Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be an analytic function germ. Then \( f \) has a decomposition of the following form:

\[
f = f_1^{m_1} \cdots f_k^{m_k}, \quad m_i \geq 1 \quad (1 \leq i \leq k), \quad k \in \mathbb{N}, \quad f_i \neq f_j \quad (i \neq j),
\]

where each \( f_i : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \), \( 1 \leq i \leq k \), is an irreducible analytic component of \( f \).

Let us define the curvature of level curves of \( f \) as follows:

\[
\hat{K}_f^C(z, w) := \begin{cases} K_f^C(z, w) & \text{if } (z, w) \in \mathbb{C}^2 \setminus f^{-1}(0) \\ K_f^C(z, w) & \text{if } (z, w) \in f_j^{-1}(0) \setminus \{0\} \quad (1 \leq j \leq k). \end{cases}
\]

The next proposition follows from Lemmas 2.9 and 2.10.

**Proposition 2.11.** \( \hat{K}_f^C \) is continuous in a punctured neighbourhood of \( 0 \in \mathbb{C}^2 \).

When we discuss the phenomenon of concentration of curvature and non concentration of curvature for a complex analytic function germ \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \), the minus sign is not essential. Accordingly we set

\[
\overline{K}_f^C(z, w) := |\hat{K}_f^C(z, w)|
\]

over a punctured neighbourhood of \( 0 \in \mathbb{C}^2 \). In §5 we shall use this definition of \( \overline{K}_f^C \) to define the notions of concentration or non concentration of curvatures in the complex case.

### 3. Tree model and real tree model

In this section we briefly review the definitions of tree model and real tree model introduced in [12] and [6, 7], respectively.

We first recall the notion of tree model. This is a kind of geometric interpretation of the classical Zariski theorem on the topology of complex plane curves ([20]).

Let \( f(x, y) \) be a complex analytic function germ of multiplicity \( m \) and mini-regular in \( x \). Let \( x = \lambda_i(y), \ i = 1, \ldots, m, \) be the complex Newton-Puiseux roots of \( f \). We denote the contact order at zero of \( \lambda_i \) and \( \lambda_j \) by

\[
O(\lambda_i, \lambda_j) := \text{ord}_0 (\lambda_i - \lambda_j)(y).
\]

Let \( h \in \mathbb{Q} \). We say that \( \lambda_i, \lambda_j \) are congruent modulo \( h^+ \) if \( O(\lambda_i, \lambda_j) > h \).

The tree model \( T(f) \) of \( f \) is defined as follows. Draw a vertical line segment as the main trunk of the tree. Mark \( m = \text{mult}_0 f(x, y) \) alongside the trunk. Let \( h_0 := \min\{O(\lambda_i, \lambda_j) \mid 1 \leq i, j \leq m\} \). Then draw a bar, \( B_0 \), on top of the main trunk. Call \( h(B_0) := h_0 \) the height of \( B_0 \). The roots are divided into equivalence classes modulo \( h_0^+ \).

We then represent each equivalence class by a vertical line segment drawn on top of \( B_0 \), and call it trunk. If a trunk consists of \( s \) roots we say it has multiplicity \( s \), and mark \( s \)
alongside. The same construction is repeated recursively. The construction terminates at the stage where all trunks contain only single, possibly multiple, roots of \( f \).

**Example 3.1.** Let \( f: (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be a polynomial function defined by
\[
f(x, y) = x(x + y)^2(x^2 - y^3)(x^3 + y^7)^3.
\]
The tree model \( T(f) \) is drawn as follows:

```
  2
   |
  3
   |
  3    3
   |
  7/3
   |
  3/2
   |
  1
```

We do not take care on the position of trunks on bars in the complex case.

We call the sets of roots corresponding to trunks *bunches*. Each bunch \( A \) is a set of roots growing through a unique bar \( B(A) \). Fix a bunch \( A \) with finite height \( h(A) \). Take a root \( \lambda_i(y) \in A \). Let \( \lambda_A(y) \) denote \( \lambda_i(y) \) with all terms \( y^e, e \geq h(A) \), omitted. Then we can write \( \lambda_i(y) \in A \) as
\[
\lambda_i(y) = \lambda_A(y) + c_i y^{h(A)} + \cdots, \quad c_i \in \mathbb{C}.
\]

We next recall the notion of real tree model. Let \( f(x, y) \) be a real analytic function germ. Consider the Newton-Puiseux roots as arcs \( x = \lambda_i(y) \) defined for \( y \in \mathbb{R}, y \geq 0 \). The complex conjugation acts on the Newton-Puiseux roots, and hence on the tree model \( T(f) \). A bunch \( A \) of \( T(f) \) is called *real* if it is stable under complex conjugation. A bar or a trunk is real if and only if so are the corresponding bunches growing on it. We denote by \( T_+(f) \) the conjugation invariant part of \( T(f) \).

Fix \( v \) a unit vector of \( \mathbb{R}^2 \). Fix any local system of coordinates \( x, y \) such that \( f(x, y) \) is mini-regular in \( x \) and \( v \) is of the form \((v_1, v_2)\) with \( v_2 > 0 \). Consider the Newton-Puiseux roots of \( f \) as arcs \( x = \lambda_i(y) \) defined for \( y \in \mathbb{R}, y \geq 0 \).

We define the real tree model of \( f \) relative to \( v \), denoted by \( RT_v(f) \), as the part of \( T_+(f) \) consisting only of those roots tangent to \( v \) with the following additional information. Let \( A \) be a real bunch such that \( B = B(A) \) is a bar of \( RT_v(f) \). Then:
- draw the trunks on \( B \) realising the sub-bunches of \( A \) keeping the clockwise order of the roots (i.e. the order of the coefficients \( c_i \) in (3.1)),
- whenever \( B \) gives a new Puiseux pair of some roots of \( A \) we mark 0 on \( B \) and draw from it the unique sub-bunch of \( A \) with \( c_i = 0 \), i.e. consisting of the roots that do not
have the new Puiseux pair at $B$. Hence we are also able to determine from the tree the sub-bunches with positive and negative $c_i$. Graphically, we identify $0 \in B$ with the point of $B$ that belongs to the trunk supporting $B$.

The real tree model $\mathbb{R}T(f)$ of $f$ is defined as follows:

- Draw a bar $B_0$ (identified with $S^1$). We define $h(B_0) = 1$ and call $B_0$ the ground bar. We mark $m(B_0) := 2 \text{mult}_0 f(x,y)$ below the ground bar. We call $m(B_0)$ the multiplicity of the ground bar.
- Draw on $B_0$ the non-trivial $\mathbb{R}T_v(f)$ for $v \in S^1$, keeping the clockwise order.
- Let $v_1, v_2$ be any two subsequent unit vectors for which $\mathbb{R}T_v(f)$ is nontrivial. Mark on $B_0$ of $\mathbb{R}T(f)$ the sign of $f$ in the sector between $v_1$ and $v_2$.

**Definition 3.2.** We call two real tree models $\mathbb{R}T(f)$, $\mathbb{R}T(g)$ isomorphic, if there is a homeomorphism $\varphi$ of their ground bars sending one tree to the other and preserving both the multiplicities and the heights of their bars, and the signs of the characteristic coefficients.

**Definition 3.3.** Given a real tree model $\mathbb{R}T(f)$ having a bar $B$ of height bigger than 1. We say that $B$ is essential, if it does not disappear under any isomorphism of real tree models.

**Example 3.4.** Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be two polynomial functions defined by

$$f(x,y) = (x - y^2)(x + y^2) \quad \text{and} \quad g(x,y) = x - y^2.$$ 

The bars of height 2 in $\mathbb{R}T(f)$ are essential, whilst those in $\mathbb{R}T(g)$ are not essential.

**Remark 3.5.** In the complex case we are drawing only the essential bars in tree models.

Blow-analytic equivalence is a notion defined by the second author in [13] as a natural equisingularity condition for real analytic function germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. We say that two real analytic function germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent, if there exist compositions of finite point blowings-up $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^2, 0)$, $\mu' : (M', \mu'^{-1}(0)) \to (\mathbb{R}^2, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (M', \mu'^{-1}(0))$ which induces a homeomorphism $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ \phi$.

**Remark 3.6.** In general blow-analytic equivalence is defined using real modifications ([13]). In the two variable case a real modification is attained by a composition of finite point blowings-up ([6]). For properties on blow-analyticity, see the surveys [3], [5].

Blow-analytic equivalence of two variable real analytic function germs is completely determined by the real tree models as follows:

**Theorem 3.7.** ([6]) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs. Then $f$ and $g$ are blow-analytically equivalent if and only if the real tree models of $f$ and $g$ are isomorphic.

**Example 3.8.** Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be two polynomial functions defined by

$$f(x,y) = x^3 - y^4, \quad g(x,y) = x^3 + y^4.$$
The real tree models $\mathbb{R}T(f)$ and $\mathbb{R}T(g)$ are drawn as follows:

Since $\mathbb{R}T(f)$ and $\mathbb{R}T(g)$ are not isomorphic, we can see by Theorem 3.7 that $f$ and $g$ are not blow-analytically equivalent.

**Definition 3.9.** We call a real analytic function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ homogeneous-like, if $f$ is blow-analytically equivalent to a homogeneous polynomial function germ.

Concerning the homogeneous-likeness of two variable real analytic function germs, we give a characterisation in terms of real tree models.

**Lemma 3.10.** For a real analytic function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, the following are equivalent.

1. $f$ is homogeneous-like.
2. Up to isomorphisms of real tree models, no bar except the ground bar appears in the real tree model $\mathbb{R}T(f)$.

**Proof.** Suppose that $f$ is homogeneous-like, namely $f$ is blow-analytically equivalent to a two variable real homogeneous polynomial function germ, and by Theorem 3.7, their real tree models are isomorphic. Since any two variable real homogeneous polynomial function germ clearly has a real tree model consisting of only the ground bar with finite trunks (including also the empty case) on it, (2) follows immediately.

On the other hand, consider a real analytic function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ having a real tree model $\mathbb{R}T(f)$ which consists of only the ground bar with finite trunks on it. Taking into account also the multiplicities of the ground bar and the trunks and signs, it is easy to construct a two variable real homogeneous polynomial function having the real tree model isomorphic to $\mathbb{R}T(f)$. Therefore (1) follows immediately from Theorem 3.7. \(\square\)

**Remark 3.11.** If $f$ is homogeneous-like, then any bar of height $> 1$ in $\mathbb{R}T(f)$ is not essential.

We say that two complex analytic function germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are topologically equivalent, if there exists a homeomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f = g \circ \phi$.

Now we recall the improved version of the Zariski theorem on the topology of two variable complex analytic function germs.
Theorem 3.12. (O. Zariski [20], Kuo - Lu [12], A. Parusiński [17]) Let \( f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be analytic function germs. Then \( f \) and \( g \) are topologically equivalent if and only if the tree models of \( f \) and \( g \) coincide.

Definition 3.13. We call a complex analytic function germ \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) homogeneous-like, if \( f \) is topologically equivalent to a homogeneous polynomial function germ.

Using Theorem 3.12 and a similar argument to Lemma 3.10, we can also characterise the homogeneous-likeness of a two variable complex analytic function germ as follows:

Lemma 3.14. For a complex analytic function germ \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \), the following are equivalent.

1. \( f \) is homogeneous-like.
2. No bar of height bigger than 1 appears in the tree model \( T(f) \).

Remark 3.15. Any real tree model has at least the ground bar. Some tree models in the complex case, however, can consist of only one trunk without any bar. This case also satisfies condition (2) in the above lemma.

4. Computation of A’Campo bumps

In [9, 10] we introduced and computed A’Campo bumps of two variable analytic function germs using infinitesimals. Here we review some notions and results which will be used to show our main results.

Take an analytic function germ \( \alpha : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0), \alpha(t) \neq 0 \).

Let \( \alpha_* := \text{Im}(\alpha) \) be the image set germ. Being an irreducible curve germ in \( \mathbb{C}^2 \), it has a unique tangent \( T(\alpha_*) \) at 0, \( T(\alpha_*) \) is a point of the Riemann Sphere \( \mathbb{C}P^1 \). We call \( \alpha_* \) an infinitesimal at \( T(\alpha_*) \). The Enriched Riemann Sphere is \( \mathbb{C}P^1_* := \{ \alpha_* \} \).

The absolute value of the curvature computed along \( \alpha_* \), if not zero, can be written as

\[
|K_C^f(\alpha(t))| = as^L + \cdots, \quad a > 0, \quad L \in \mathbb{Q},
\]

where \( s = s(t) \) is the arc length. This is dominated by the leading term \( as^L \) as \( s \to 0 \). If \( |K_C^f| \equiv 0 \) along \( \alpha_* \), we write \( (a, L) := (0, \infty) \). Hence we introduce the notations

\[
(a, L) := a\delta^L, \quad 0_\nu := 0\delta^\infty, \quad \mathcal{V}(\mathbb{R}) := \{ a\delta^L \mid a \neq 0 \} \cup \{ 0_\nu \},
\]

where \( \delta \) is a symbol.

A lexicographic ordering on \( \mathcal{V}(\mathbb{R}) \) is defined: \( 0_\nu \) is the smallest element, and

\[
a\delta^L > a'\delta^{L'}, \quad \text{if and only if either} \quad L < L', \quad \text{or else} \quad L = L', \quad a > a'.
\]

The curvature function \( K_* \) on \( \mathbb{C}P^1_* \), and the component \( L_* \) are defined as follows:

\[
K_* : \mathbb{C}P^1_* \to \mathcal{V}(\mathbb{R}), \quad \alpha_* \to a\delta^L; \quad L_* : \mathbb{C}P^1_* \to \mathbb{Q}, \quad \alpha_* \to L.
\]

Recall the field \( \mathbb{F} \) of convergent fractional power series in an indeterminate \( y \) is algebraically closed. A non-zero element of \( \mathbb{F} \) is a convergent series

\[
\alpha(y) = a_0y^{n_0/N} + \cdots + a_iy^{n_i/N} + \cdots, \quad n_0 < n_1 < \cdots; \quad n_i \in \mathbb{Z},
\]
where $0 \neq a_i \in \mathbb{C}$, $N \in \mathbb{Z}^+$, $GCD(N, n_0, n_1, \ldots) = 1$. The conjugates of $\alpha$ are
\[ o_{\text{conj}}^{(k)}(y) := \sum a_i \theta^k n_i y^{n_i/N}, \quad 0 \leq k \leq N - 1, \quad \theta := e^{2\pi i/N}. \]
The order is $O_y(\alpha) := n_0/N$, $O_y(0) := +\infty$. The Puiseux multiplicity is $m_{\text{puis}}(\alpha) := N$.

The following are integral domains:

$\mathcal{D}_1 := \{ \alpha \in \mathbb{F} \mid O_y(\alpha) \geq 1 \}, \quad \mathcal{D}_1^+ := \{ \alpha \mid O_y(\alpha) > 1 \},$

having quotient field $\mathbb{F}$.

As in Projective Geometry, $\mathbb{C}P_1^1$ is a union of two charts: $\mathbb{C}P_1^1 = \mathbb{C}_* \cup \mathbb{C}'$;
\[ \mathbb{C}_* := \{ \beta \in \mathbb{C}P_1^1 \mid T(\beta) \neq [1 : 0] \}, \quad \mathbb{C}' := \{ \beta \in \mathbb{C}P_1^1 \mid T(\beta) \neq [0 : 1] \}. \]

Next we define the contact order $C_{\text{ord}}(\alpha_*, \beta_*)$. We can assume $\alpha_*, \beta_* \in \mathbb{C}_*$. Then
\[ C_{\text{ord}}(\alpha_*, \beta_*) := \begin{cases} \infty & \text{if } \alpha_* = \beta_*, \\ \max_{i,j} \{ O_y(\alpha_{\text{conj}}^{(i)}(y) - \beta_{\text{conj}}^{(j)}(y)) \} & \text{if } \alpha_* \neq \beta_. \end{cases} \]

The horn subspaces of $\mathbb{C}P_1^1$ centered at $\alpha_*$ of order $e, e'$ are
\[ \mathcal{H}_e(\alpha_*) := \{ \beta_* \mid C_{\text{ord}}(\alpha_*, \beta_*) \geq e \}, \quad \mathcal{H}_{e+}(\alpha_*) := \{ \beta_* \mid C_{\text{ord}}(\alpha_*, \beta_*) > e \}, \]
respectively. In particular, $C_{\text{ord}}(\alpha_*, \beta_*) = 1$ if $T(\alpha_*) \neq T(\beta_*)$; and $\mathcal{H}_1(\alpha_*) = \mathbb{C}P_1^1$ for all $\alpha_*$. When there is no need to specify $\alpha_*$, we write $\mathcal{H}_e := \mathcal{H}_e(\alpha_*)$.

Let $\mathcal{H}_e(\alpha_*)$ be given. If $\eta(y) = \alpha(y) + [cy^e + \cdots]$, $c$ a generic number, then $L_s(\eta_s)$ is a constant. We write this constant as $L_s(\mathcal{H}_e^{\text{grc}}(\alpha_*)$. A horn interval of radius $r$, $r > 0$, is, by definition,
\[ \mathcal{H}_e(\alpha_*, r) := \{ \beta_* \mid \beta(y) = \alpha(y) + (cy^e + \cdots), \mid c \mid \leq r \}. \]

**Definition 4.1.** A horn subspace $\mathcal{H}_e(\alpha_*)$ is a curvature tableland if

1. $\beta_* \in \mathcal{H}_e(\alpha_*) \implies L_s(\beta_*) \geq L_s(\mathcal{H}_e^{\text{grc}}(\alpha_*)$;
2. in the case $e > 1$, there exists $e', 1 \leq e' < e$, such that
\[ \nu_* \in \mathcal{H}_{e'}(\alpha_*) \setminus \mathcal{H}_e(\alpha_*) \implies L_s(\nu_*) > L_s(\mathcal{H}_e^{\text{grc}}(\alpha_*)). \]

**Example 4.2.** Let $f(x, y) = x^2 - y^5$. Then we have
\[ f_x = 2x, \quad f_y = -5y^4, \quad f_{xx} = 2, \quad f_{xy} = 0, \quad f_{yy} = -20y^3. \]

Therefore we have
\[ K_f(x, y) = \frac{200|y|^6|8x^3 - 5y^5|^2}{(4|x|^2 + 25|y|8)^3}. \]

We consider the horn subspace $\mathcal{H}_4(0_*).$ Then we can see
\[ L_s(\mathcal{H}_4^{\text{grc}}(0_*)) = -8 = L_s(\beta_*) \quad \text{for all } \beta_* \in \mathcal{H}_4(0_*). \]

Let $0 < \varepsilon < \frac{3}{2}$. If $\nu_* \in \mathcal{H}_{4-\varepsilon}(0_*) \setminus \mathcal{H}_4(0_*), \quad \text{then}$
\[ -8 < L_s(\nu_*) \leq -8 + 6\varepsilon. \]

It follows that $L_s(\nu_*) > L_s(\mathcal{H}_4^{\text{grc}}(0_*)).$ Thus $\mathcal{H}_4(0_*)$ is a curvature tableland.
**Definition 4.3.** Let $\mathcal{H}_e$ be a curvature tableland. Take $\beta_* \in \mathcal{H}_e$. We say $K_\star$ has an A’Campo bump on $\mathcal{H}_e^+(\beta_*)$, or simply say $\mathcal{H}_e^+(\beta_*)$ is an A’Campo bump, if there exists $\epsilon > 0$ such that

$$
\mu_\star \in \mathcal{H}_e(\beta_*, \epsilon) \implies K_\star(\beta_*) \geq K_\star(\mu_\star).
$$

Let us apply a generic unitary transformation so that $f$ is mini-regular in $z$ where $m = O(f)$. Then the initial form $H_m$ is written as follows:

$$
H_m(z, w) = c(z - c_1 w)^{m_1} \cdots (z - c_r w)^{m_r}; \quad m_i \geq 1; \quad c_i \neq c_j; \quad \text{if}; \quad i \neq j,
$$

where $1 \leq r \leq m$, $\sum m_i = m$, $c \neq 0$. Thus $H_m(z, w)$ is degenerate if and only if $r < m$.

Let $\zeta_i$ denote the Newton-Puiseux roots of $f(z, w)$, and $\gamma_j$ those of $f_z$:

$$
f(z, w) = \text{unit} \cdot \prod_{i=1}^{m}(z - \zeta_i(w)), \quad f_z(z, w) = \text{unit} \cdot \prod_{j=1}^{m-1}(z - \gamma_j(w)),
$$

where $O_w(\zeta_i), O_w(\gamma_j) \geq 1$. Each $\gamma_j$ is called a polar, and so is $\gamma_{j\star}$.

**Definition 4.4.** Given a polar $\gamma$, let $d_{gr}(\gamma)$ denote the smallest number $e$ such that

$$
O_w(\|\text{Grad } f(\gamma(w), w)\|) = O_w(\|\text{Grad } f(\gamma(w) + uw^e, w)\|),
$$

where $u \in \mathbb{C}$ is a generic number. We call $d_{gr}(\gamma)$ the gradient order of $\gamma$.

The $D$-gradient canyon of $\gamma$, and the $*$-gradient canyon of $\gamma_\star$ are

$$
G(\gamma) := \{ \alpha \in D_1 \mid O_y(\alpha - \gamma) \geq d \}, \quad G_\star(\gamma_\star) := \mathcal{H}_d(\gamma_\star), \quad d := d_{gr}(\gamma),
$$

respectively. When there is no confusion, we call them “canyons”; we also write

$$
d := d_{gr}(\gamma), \quad G := G(\gamma), \quad G_\star := G_\star(\gamma_\star).
$$

The degree and multiplicity of $G, G_\star$ are, respectively,

$$
d_{gr}(G) := d_{gr}(G_\star) := d, \quad m(G) := \sharp \{ k \mid G(\gamma_k) = G \}, \quad m(G_\star) := \sharp \{ k \mid G_\star(\gamma_k) = G_\star \}.
$$

Finally, we say $G(\gamma)$ and $G_\star(\gamma_\star)$ are minimal if

$$
G(\gamma_j) \subseteq G(\gamma) \implies G(\gamma_j) = G(\gamma).
$$

Let $\gamma$ be a polar with $d < \infty$. We now define $L_\gamma \in \mathbb{Q}$, and a rational function $R_\gamma(u), u \in \mathbb{C}$. We can assume $\gamma \in D_{1+}$ so that $T(\gamma_\star) = [0 : 1]$. If $d > 1$, define $L_\gamma, R_\gamma(u)$ by

$$
|K^C_f(\gamma(y) + uy^d, y)| := 2R_\gamma(u)y^{2L_\gamma} + \cdots, \quad R_\gamma(u) \neq 0,
$$

where $y$ can be considered as the arc length of $(\gamma(y) + uy^d)_\star$, since $\lim y/s = 1$.

In the case $d = 1$, define $L_\gamma$ and $R_\gamma(u)$ by

$$
|K^C_f(\gamma(y) + \frac{uy}{\sqrt{1 + |u|^2}} - \frac{y}{\sqrt{1 + |u|^2}})| := 2R_\gamma(u)y^{2L_\gamma} + \cdots, \quad R_\gamma(u) \neq 0.
$$

**Theorem 4.5.** ([9]) A minimal $*$-canyon with $d < \infty$ is a curvature tableland, and vice versa.

Take a minimal $G(\gamma), d < \infty$. Take a local maximum $R_\gamma(c)$ of $R_\gamma(u)$. Then $\mathcal{H}_{d+(\gamma_\star^+)}$ is an A’Campo bump, where $\gamma_\star^+(y) := \gamma(y) + cy^d$.

All A’Campo bumps can be found in this way.
Addendum 4.6. ([9]) Every $\mathcal{G}(\gamma)$ with $d > 1$ is minimal. In this case, 

$$
m(\mathcal{G}) = \sharp \{ k \mid \gamma_k \in \mathcal{G} \}, \quad m(\mathcal{G}_*) = \sharp \{ k \mid \gamma_{k*} \in \mathcal{G}_* \}.
$$

Let $r$ be as in (4.1). There are exactly $r - 1$ polars of gradient degree 1; moreover, 

$$
d_{gr}(\gamma) = 1 \implies \mathcal{G}(\gamma) = \mathbb{D}_1, \quad m(\mathcal{G}(\gamma)) = r - 1.
$$

A minimal $\mathcal{G}(\gamma)$ with $d = 1$ exists if and only if $H_m(z, w)$ is non-degenerate. In this case every polar has $d = 1$, $\mathbb{C}P^1$ is the only curvature tableland and the only $*$-canyon.

Next we recall the Newton polygon relative to a polar. Let $\gamma$ be a given polar, not a multiple root of $f(z, w)$, i.e., $f(\gamma(w), w) \neq 0$. We can apply a unitary transformation, if necessary, so that $T(\gamma_*) = [0 : 1]$, $\gamma \in \mathbb{D}_1$.

Let us change coordinates:

$$
Z := z - \gamma(w), \quad W := w, \quad F(Z, W) := f(Z + \gamma(W), W).
$$

Then

$$
\Delta_f(z, w) = \Delta_F(Z, W) + \gamma''(W)F^2, \quad \|\text{Grad}_{z,w}f\| \approx \|\text{Grad}_{Z,W}F\|.
$$

Recall that a monomial term $aZ^iW^q$, $a \neq 0, q \in \mathbb{Q}$, is represented by a "Newton dot" $\gamma$ at $(i, q)$. We shall simply say $(i, q)$ is a dot.

If $i \geq 1$, then $(i, q)$ is a dot of $F(Z, W)$ if and only if $(i - 1, q)$ is one of $F_Z$. Since $\gamma$ is a polar, $F_Z$ has no dot of the form $(0, q)$; $F(Z, W)$ has no dot of the form $(1, q)$.

As $f(\gamma(w), w) \neq 0$, we know $F(0, W) \neq 0$. Hence

$$
F(0, W) := aW^h + \cdots, \quad a \neq 0, \quad h = O_W(F(0, W)),
$$

and then $(0, h)$ is a vertex of the Newton polygon $\mathcal{NP}(F)$, $(0, h - 1)$ is one of the Newton polygon $\mathcal{NP}(F_W)$.

Let $E_{\text{top}}$ denote the top edge of the Newton polygon $\mathcal{NP}(F)$, i.e., the edge with left vertex $(0, h)$. Let $(m_{\text{top}}, q_{\text{top}})$ denote the right vertex of $E_{\text{top}}$, and $\theta_{\text{top}}$ the angle of $E_{\text{top}}$.

$$
\tan \theta_{\text{top}} = \text{co-slope of } E_{\text{top}},
$$

where the co-slope of a line passing through $(x, 0)$ and $(0, y)$ is, by definition, $y/x$.

Let $(m'_{\text{top}}, q'_{\text{top}}) \neq (0, h)$ be the dot of $F$ on $E_{\text{top}}$ which is closest to $(0, h)$. Then, clearly,

$$
2 \leq m'_{\text{top}} \leq m_{\text{top}}, \quad \frac{h - q'_{\text{top}}}{m'_{\text{top}}} = \frac{h - q_{\text{top}}}{m_{\text{top}}} = \tan \theta_{\text{top}}.
$$

Lemma 4.7. ([9]) Let $\mathcal{L}^*$ denote the line joining $(0, h - 1)$ (which is not a dot of $F_Z$) and a dot of $F_Z$ such that no dot of $F_Z$ lies below $\mathcal{L}^*$. Let $\sigma^*$ denote the co-slope of $\mathcal{L}^*$. Then

$$
\tan d_{gr}(\gamma) = \sigma^* \geq \tan \theta_{\text{top}},
$$

where $\sigma^* = \tan \theta_{\text{top}}$ if and only if $\sigma^* = 1$.

In order to show the main result in the complex case (Theorem 5.4), we shall use the above Theorem 4.5, Addendum 4.6 and Lemma 4.7.
5. Characterisations of no concentration of curvature

Let \( K = \mathbb{R} \) or \( \mathbb{C} \). Consider the families of convergent Puiseux demi-arcs on \( \mathbb{R}^2 \) and the families of convergent Puiseux arcs on \( \mathbb{C}^2 \)

\[
x = a_0 y + a_1 y^{\frac{n_1}{N}} + \cdots + a_i y^{\frac{n_i}{N}} + \cdots,
\]

or

\[
y = b_1 x^{\frac{n_1}{N}} + \cdots + b_i x^{\frac{n_i}{N}} + \cdots,
\]

where \( 1 < \frac{n_1}{N} < \frac{n_2}{N} \cdots \), \( a_i, b_i \in \mathbb{K} \), \( n_i, N \in \mathbb{Z}^+ \). Here the demi-arc means that they are defined for \( y \in \mathbb{R}, y \geq 0 \) or \( x \in \mathbb{R}, x \geq 0 \). Let us denote this family by \( \mathcal{A} \).

Let \( f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0) \) be an analytic function germ not identically zero, mini-regular in \( x \). For \( \gamma \in \mathcal{A} \), we define the curvature exponent along \( \gamma \) by

\[
e(\gamma) := O(\overline{K}_f(\gamma)).
\]

Then we define the curvature exponent of \( f \) by

\[
\hat{e}(f) := \inf_{\gamma \in \mathcal{A}} \{e(\gamma)\}.
\]

**Remark 5.1.** We note that

\[-\hat{e}(f) = \min \{s \mid (x, y)|^s|\overline{\mathcal{K}}_f| \lesssim 1\}.
\]

Therefore \( \hat{e}(f) \) is attained by some arc in \( \mathcal{A} \).

Now we define the notion of concentration of curvature. Let \( S := \{v \in \mathbb{K}^2 \mid \|v\| = 1\} \).

**Definition 5.2.** (1) We say that \( f \) has concentration of curvature at \( 0 \in \mathbb{K}^2 \), if the set

\[
\{v \in S \mid \exists \gamma \in \mathcal{A} \text{ s.t. } \lim_{t \to 0} \frac{\gamma(t)}{\|\gamma(t)\|} = v \& e(\gamma) = \hat{e}(f)\}
\]

is finite.

(2) We say that \( f \) has no concentration of curvature at \( 0 \in \mathbb{K}^2 \), if \( f \) does not have concentration of curvature at \( 0 \in \mathbb{K}^2 \).

**Lemma 5.3.** Let \( f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0) \) be a homogeneous-like analytic function germ. Then \( f \) has no concentration of curvature.

**Proof.** We show only the real case. The complex case follows similarly.

Since \( f \) is homogeneous-like, it follows from Theorem 3.7 that there is a homogeneous polynomial \( H : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) such that \( \mathbb{R}T(f) \) is isomorphic to \( \mathbb{R}T(H) \). Therefore \( f \) has a decomposition of the following form:

\[
f = f_1^{m_1} \cdots f_k^{m_k} h, \ m_i \geq 0 \ (1 \leq i \leq k), \ k \in \mathbb{N} \cup \{0\},
\]

where \( f_i(x, y) = x - c_i y + \phi_i(x, y) \) with \( j^1 \phi_i(0, 0) = 0 \), and \( c_1 < c_2 < \cdots < c_k \), and either \( In(h)^{-1}(0) = \{0\} \) as germs at \( 0 \in \mathbb{R}^2 \) or \( h \) is a unit. Here \( In(h) \) means the initial homogeneous form of \( h \).

Let \( \Delta_f \) be the denominator of \( \tilde{K}_f \) after cancellation of \( f_1^{3(m_1-1)} \cdots f_k^{3(m_k-1)} \) as in subsection 2. If \( \Delta_f \equiv 0 \), then non concentration of curvature happens.

We next consider the directions \((e, 1), \ e \in \mathbb{R}, \) and \((1, 0)\). Note that if \( \tilde{\Delta}_f \) is not identically 0, then there are only finitely many \( e \)'s such that \( In(\Delta)(e, 1) = 0 \).
• In the case where \( \text{In}(\tilde{\Delta}_f)(e, 1) \neq 0 \), \( K_{R_f} \approx \frac{1}{|y|} \) along any convergent Puiseux arc \( \alpha_a \) with \( a_0 = e \).
• In the case where \( \text{In}(\tilde{\Delta}_f)(e, 1) = 0 \), \( K_{R_f} \ll \frac{1}{|y|} \) along any convergent Puiseux arc \( \alpha_a \) with \( a_0 = e \).

Similarly for convergent Puiseux arcs with direction \((1, 0)\), we can see the following:
• In the case where \( \text{In}(\tilde{\Delta}_f)(1, 0) \neq 0 \), \( K_{R_f} \approx \frac{1}{|x|} \) for any \( \alpha_{(0,b)} \).
• In the case where \( \text{In}(\tilde{\Delta}_f)(1, 0) = 0 \), \( K_{R_f} \ll \frac{1}{|x|} \) for any \( \alpha_{(0,b)} \).

Therefore non concentration of curvature happens.

Using the above lemmas, we give some characterisations of the non concentration of curvature in the complex case.

**Theorem 5.4.** For an analytic function germ \( f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0) \) not identically zero, the following are equivalent.

1. \( f \) is homogeneous-like.
2. No bar of height bigger than 1 appears in the tree model \( T(f) \).
3. \( f \) has no concentration of curvature at \( 0 \in \mathbb{C}^2 \).

**Proof.** By Lemmas 3.14 and 5.3, it suffices to show (3) implies (1). Taking a unitary transformation if necessary, we may assume that \( f \) is mini-regular in \( z \).

Suppose that there exists a bar of height bigger than 1 in the tree model \( T(f) \). Then it follows from the Kuo-Lu theorem ([12]) that there is a polar \( \gamma \) which leaves a trunk on this bar. Consider the Newton polygon relative to that polar. The top edge \( E_{\text{top}} \) has co-slope bigger than 1, because the co-slope is the height of the bar. Hence, by Lemma 4.7, the gradient order \( d \) of \( \gamma \) is bigger than 1. By Addendum 4.6 and Theorem 4.5, \( G(r) \) with \( d > 1 \) is minimal and \( G(\gamma) \) is a curvature tableland. If \( f \) has another polar \( \gamma' \) with gradient order \( d' = 1 \), \( G(\gamma') \) with \( d' = 1 \) is not minimal by Addendum 4.6. It follows from Theorem 4.5 that \( G(\gamma') \) is not a curvature tableland. Thus \( f \) has concentration of curvature at \( 0 \in \mathbb{C}^2 \). \( \square \)

The curvature of a complex analytic function itself can change under a non-unitary linear transformation. Nevertheless we can note the following.

**Corollary 5.5.** For the two variable complex analytic function germs, the appearance of concentration of curvature is a topological invariant.

By Lemmas 3.10 and 5.3 we have the following result in the real case.

**Theorem 5.6.** For an analytic function germ \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0) \) not identically zero, the following are equivalent.

1. \( f \) is homogeneous-like.
2. Up to isomorphisms of real tree models, no bar except the ground bar appears in the real tree model \( R_T(f) \).
3. \( f \) has no concentration of curvature at \( 0 \in \mathbb{R}^2 \).

In the real case non concentration of curvature does not always imply the homogeneous-likeness. We give an example to demonstrate it. In order to see it, we prepare some
notations. Let \( g, h : [0, \delta) \rightarrow \mathbb{R} \) be convergent fractional power series functions, where \( h \) is not identically zero. Then we write \( g \sim h \) if \( \frac{g(y)}{h(y)} \) tends to 1 as \( y \to +0 \). If \( g(y) - h(y) \) consists of terms with higher orders than the order of \( h(y) \), we write \( g(y) = h(y) + \text{HOT} \). Note that this \( \text{HOT} \) is not the high order terms in the usual sense.

**Proposition 5.7.** Let \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0) \) be a polynomial function defined by

\[
f(x, y) = (x - y^2)^3 + x^2y^6 - xy^8 + y^{12}.
\]

Then the non-directed curvature \( \overline{K}_f^R \) tends 2 along any convergent Puiseux demi-arc \( \alpha_a \) and \( \alpha_{(0,b)} \) on \( \mathbb{R}^2 \) as \( y \to +0 \) and \( x \to +0 \) respectively, namely non concentration of curvature happens with a constant coefficient. On the other hand, \( f \) is not homogeneous-like.

**Proof.** Let us express \( \alpha_a \) and \( \alpha_{(0,b)} \) as follows:

\[
\alpha_a : \quad x = a_0 y^{s_0} + a_1 y^{s_1} + a_2 y^{s_2} + \cdots, \quad 1 \leq s_0 < s_1 < s_2 < \cdots,
\]

\[
\alpha_{(0,b)} : \quad y = b_1 x^{s_1} + b_2 x^{s_2} + \cdots, \quad 1 < s_1 < s_2 < \cdots.
\]

We set \( G(x, y) := (f_x(x, y)^2 + f_y(x, y)^2)^{3/2} \).

By an easy computation, we have

\[
f_x = 3(x - y^2)^2 + 2xy^6 - y^8,
\]

\[
f_y = -6y(x - y^2)^2 + 6x^2y^5 - 8xy^7 + 12y^{11},
\]

\[
f_{xx} = 6(x - y^2) + 2y^6,
\]

\[
f_{xy} = -12y(x - y^2) + 12x^5 - 8y^7,
\]

\[
f_{yy} = -6(x - y^2)^2 + 24y^4(x - y^2) + 30x^2y^4 - 56xy^6 + 132y^{10}.
\]

Then we have

\[
\Delta_f = -(-6y(x - y^2)^2 + 6x^2y^5 - 8xy^7 + 12y^{11})^2 (6(x - y^2) + 2y^6)
\]

\[
-3(3(x - y^2)^2 + 2xy^6 - y^8)(-6y(x - y^2)^2 + 24y^2(x - y^2) + 30x^2y^4 - 56xy^6 + 132y^{10})
\]

\[
+2(3(x - y^2)^2 + 2xy^6 - y^8)(-6y(x - y^2)^2 + 24y^2(x - y^2) + 30x^2y^4 - 56xy^6 + 132y^{10})
\]

\[
G = \{(3(x - y^2)^2 + 2xy^6 - y^8)^2 + (-6y(x - y^2)^2 + 24y^2(x - y^2) + 30x^2y^4 - 56xy^6 + 132y^{10})^2\}^{3/2}.
\]

In order to see that non concentration of curvature happens, we have to know the order of \( \overline{K}_f^R \) in \( y \) and \( x \) along \( \alpha_a \) and \( \alpha_{(0,b)} \), respectively. We first compute the order of \( \overline{K}_f^R \) along \( \alpha_a \). If there is no cancellation of the coefficients of the lowest order terms of \( -f_y f_{xx} - f_x f_{yy} \) and \( 2f_x f_y f_{xy} \), it is enough to consider the orders and coefficients of the lowest order terms of \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \). But if the cancellation happens, we have to pay attention to more terms of them. We divide the situation into three cases. In the first two cases the cancellation of the coefficients of the lowest order terms does not happen, but such a cancellation happens in the third case.

Case (I;\( \alpha_a \)) : \( 1 \leq s_0 < 2, a_0 \neq 0 \).

Along \( \alpha_a \) we have

\[
f_x \sim 3a_0^2 y^{2s_0}, \quad f_y \sim -6a_0^2 y^{2s_0+1}, \quad f_{xx} \sim 6a_0 y^{s_0}, \quad f_{xy} \sim -12a_0 y^{s_0+1}, \quad f_{yy} \sim 6a_0 y^{2s_0}.
\]
Therefore we have
\[-f_x^2 f_{yy} \sim 54a_0^6 y^{6s_0}, \quad -f_x^2 f_{xx} \sim -216a_0^5 y^{5s_0+2}, \quad 2f_x f_y f_{xy} \sim 432a_0^5 y^{5s_0+2},\]
and
\[\Delta f \sim 54a_0^6 y^{6s_0}, \quad G \sim 27a_0^6 y^{6s_0}\]
along \(\alpha_a\). It follows that \(\bar{K}_f^{\mathbb{R}} = \frac{|\Delta f|}{G}\) tends to 2 along \(\alpha_a\) as \(y \to +0\).

Case (II; \(\alpha_a\)) : \(s_0 = 2, a_0 \neq 1\).
Along \(\alpha_a\) we have
\[f_x \sim 3(a_0 - 1)^2 y^4, \quad f_y \sim -6(a_0 - 1)^2 y^5, \quad f_{xx} \sim 6(a_0 - 1) y^2, \quad f_{xy} \sim -12(a_0 - 1)^2 y^7, \quad f_{yy} \sim -6(a_0 - 1)(a_0 - 5) y^4.\]
Therefore we have
\[-f_x^2 f_{yy} \sim 54(a_0 - 1)^5(a_0 - 5) y^{12}, \quad -f_y^2 f_{xx} \sim -216(a_0 - 1)^5 y^{12}, \quad 2f_x f_y f_{xy} \sim 432(a_0 - 1)^5 y^{12},\]
and
\[\Delta f \sim 54(a_0 - 1)^6 y^{12}, \quad G \sim 27(a_0 - 1)^6 y^{12}\]
along \(\alpha_a\). It follows that \(\bar{K}_f^{\mathbb{R}}\) tends to 2 along \(\alpha_a\) as \(y \to +0\).

Case (III; \(\alpha_a\)) : \(s_0 = 2, a_0 = 1\).
In this case the cancellation of the coefficients mentioned above happens. We set
\[A := a_1 y^{s_1} + a_2 y^{s_2} + \cdots.\]
We further divide this case into three cases.

(1) Case (III-1) : \(2 < s_1 < 4, a_1 \neq 0\).
Along \(\alpha_a\) we have
\[f_x = 3A^2 + y^8 + \text{HOT}, \quad f_y = -6A^2 y - 2y^9 + \text{HOT} = -2y(3A^2 + y^8) + \text{HOT}, \quad f_{xx} = 6A + 2y^6 + \text{HOT}, \quad f_{xy} = -12Ay + 4y^7 + \text{HOT}, \quad f_{yy} = -6A^2 + 24Ay^2 - 26y^8 + \text{HOT}.\]
Therefore we have
\[-f_x^2 f_{yy} = (6A^2 - 24Ay^2 + 26y^8)(3A^2 + y^8)^2 + \text{HOT}, \quad -f_y^2 f_{xx} = -(24Ay^2 + 8y^8)(3A^2 + y^8)^2 + \text{HOT}, \quad 2f_x f_y f_{xy} = (48Ay^2 - 16y^8)(3A^2 + y^8)^2 + \text{HOT},\]
and
\[\Delta f \sim (6A^2 + 2y^8)(3A^2 + y^8)^2 \sim 54a_1^6 y^{6s_1}, \quad G \sim 27a_1^6 y^{6s_1}\]
along \(\alpha_a\). It follows that \(\bar{K}_f^{\mathbb{R}}\) tends to 2 along \(\alpha_a\) as \(y \to +0\).

(2) Case (III-2) : \(s_1 = 4, a_1 \neq 0\).
Along \(\alpha_a\) we have
\[f_x \sim (3a_1^2 + 1)y^8, \quad f_y \sim -2(3a_1^2 + 1)y^9, \quad f_{xx} = 6A + 2y^6 + \text{HOT}, \quad f_{xy} = -12Ay + 4y^7 + \text{HOT}, \quad f_{yy} = -6a_1^2 y^8 + 24Ay^2 - 26y^8 + \text{HOT}.\]
Therefore we have
along $\alpha$. It follows that $K_f^R$ tends to 2 along $\alpha_a$ as $y \to +0$.

(2) Case (III-3) : $s_1 > 4$.

Along $\alpha_a$ we have
\[ f_x \sim y^8, \quad f_y \sim -2y^9, \quad f_{xx} = 6A + 2y^6 + \text{HOT}, \]
\[ f_{xy} = -12Ay + 4y^7 + \text{HOT}, \quad f_{yy} = 24Ay^2 - 26y^8 + \text{HOT}. \]

Therefore we have
\[ -f_x^2f_{yy} = (6A^2 + 2y^6)(3a_1^2 + 1)^2y^{18} + \text{HOT}, \]
\[ f_{yy}^2 = -(24A + 8y^6)(3a_1^2 + 1)^2y^{18} + \text{HOT}, \]
\[ 2f_xf_yf_{xy} = (48A - 16y^6)(3a_1^2 + 1)^2y^{18} + \text{HOT}, \]

and
\[ \Delta_f \sim 2(3a_1^2 + 1)^3y^{24}, \quad G \sim (3a_1^2 + 1)^3y^{24} \]
along $\alpha_a$. It follows that $K_f^R$ tends to 2 along $\alpha_a$ as $y \to +0$.

We next compute the order of $K_f^R$ along $\alpha_{(0,b)}$. In the case where $b_1 = 0$, namely $y$ is identically zero, we have
\[ f_x = 3x^2, \quad f_y = 0, \quad f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = -6x^2 \]
along $\alpha_{(0,b)}$. Therefore we have
\[ -f_x^2f_{yy} = 54x^6, \quad -f_y^2f_{xx} = 0, \quad 2f_xf_yf_{xy} = 0, \]

and
\[ \Delta_f = 54x^6, \quad G = 27x^6 \]
along $\alpha_{(0,b)}$. It follows that $K_f^R$ tends to 2 along $\alpha_{(0,b)}$ as $x \to +0$.

In the case where $b_1 \neq 0$, we have
\[ f_x \sim 3x^2, \quad f_y \sim -6b_1 x^{s_1+2}, \quad f_{xx} \sim 6x, \quad f_{xy} \sim -12b_1 x^{s_1+1}, \quad f_{yy} \sim -6x^2 \]
along $\alpha_{(0,b)}$. Therefore we have
\[ -f_x^2f_{yy} = 54x^6, \quad -f_y^2f_{xx} = -216b_1^2 x^{2s_1+5}, \quad 2f_xf_yf_{xy} = 432b_1^2 x^{2s_1+5}, \]

and
\[ \Delta_f \sim 54x^6, \quad G \sim 27x^6 \]
along $\alpha_{(0,b)}$. It follows that $K_f^R$ tends to 2 along $\alpha_{(0,b)}$ as $x \to +0$.

Lastly we show that $f$ is not homogeneous-like. After taking an analytic transformation of $\mathbb{R}^2$ at $0 \in \mathbb{R}^2, x = X^2 + Y$, $y = Y$, $f$ has the following form:
\[ f(X,Y) = X^3 + X^2Y^6 + XY^8 + Y^{12}. \]
Then we consider the family of polynomials $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $t \in I = [0, 1]$, defined by

$$f_t(x, y) = x^3 + x^2y^6 + txy^8 + y^{12}.$$ 

For any $t \in I$, the weighted initial form of $f_t$ with respect to the system of weights $(\frac{1}{3}, \frac{1}{12})$ is

$$g_t(x, y) = x^3 + txy^8 + y^{12},$$

and $g_t$ has an isolated singularity at $0 \in \mathbb{R}^2$. Therefore, by the blow-analytic triviality theorem in [4] (or [2]), we can see that $f$ is blow-analytically equivalent to $h(x, y) = x^3 + y^{12}$. Since $h$ clearly has an essential bar of height 4, it follows from Theorem 3.7 and Remark 3.11 that $f$ is not homogeneous-like.

□

References


Department of Mathematics, Hyogo University of Teacher’s Education, Hyogo, Japan

E-mail address: koike@hyogo-u.ac.jp

School of Mathematics, University of Sydney, Sydney, NSW, 2006, Australia

E-mail address: tck@maths.usyd.edu.au

School of Mathematics, University of Sydney, Sydney, NSW, 2006, Australia

E-mail address: laurent@maths.usyd.edu.au