NOTES ON (SSP) SETS

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Abstract. In [8] we investigate the directional behaviour of bi-Lipschitz homeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ for which there exist the limits $\lim_{n\to\infty} nh(\frac{x}{n})$, denoted by $\overline{h}(x)$. The existence of such $\overline{h}(x)$ makes trivial to see that $\overline{h}(D(A)) = D(h(A))$ for arbitrary set-germs A at $0 \in \mathbb{R}^n$.

Recently, J. Edson Sampaio made the remarkable observation ([9]) that we always can assume the existence of a subsequence $n_i \in \mathbb{N}$, such that $\lim_{n_i\to\infty} n_i h(\frac{x}{n_i}) = dh(x)$ (in his notation) and this dh, although not so strong as $\overline{h}$, behaves as well directional-wise for subanalytic sets. He uses this fact to show that bi-Lipschitz homeomorphic subanalytic sets have bi-Lipschitz homeomorphic tangent cones.

The purpose of this note is to show that Sampaio's dh works as well for (SSP) sets, that is, the above result is characteristic for (SSP) sets, a much wider class. In particular we show that the transversality between (SSP) sets is preserved under bi-Lipschitz homeomorphisms (see 2.17).

1. Introduction.

In [5] we proved that the dimension of the common direction set of two subanalytic subsets is a bi-Lipschitz invariant. In proving that, we introduced and essentially used the notion of sequence selection property, denoted by (SSP) for short. Subsequently we have published three more papers [6], [7] and [8], where we proved essential directional properties of sets satisfying (SSP) with respect to bi-Lipschitz homeomorphisms. For instance we proved two types of (SSP) structure preserving theorems, and we introduced the notion of directional homeomorphism, proving a unified (SSP) structure preserving theorem with directional homeomorphisms.

In this note, using Sampoio's idea, we generalise his main result in [9] and the aforementioned main result in [5] to the case of the (SSP) setting. Although the proofs are in the spirit of [8] , basically the same as in [9] (at times even simpler), due to the wide potential applications, we believe that it is still worth mentioning this generalisation.

We describe both the notions and notations necessary for this topic and our results in the (SSP) setting in the next section.

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2. Directional Properties of Sets

In this section we recall the notions of direction set and sequence selection property, and also several elementary properties concerning (SSP).

2.1. **Direction set.** Let us recall the notion of direction set.

Definition 2.1. Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We define the *direction set* $D(A)$ of A at $0 \in \mathbb{R}^n$ by

$$D(A) := \{a \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{0\},\ x_i \to 0 \in \mathbb{R}^n \text{ s.t. } \frac{x_i}{\|x_i\|} \to a,\ i \to \infty\}.$$

Here S^{n-1} denotes the unit sphere centred at $0 \in \mathbb{R}^n$.

For a subset $A \subset S^{n-1}$, we denote by $L(A)$ a half-cone of A with the origin $0 \in \mathbb{R}^n$ as the vertex:

$$L(A) := \{ta \in \mathbb{R}^n \mid a \in A,\ t \geq 0\}.$$

For a set-germ A at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, we put $LD(A) := L(D(A))$, and call it the *real tangent cone* of A at $0 \in \mathbb{R}^n$.

2.2. **Sequence selection property.** Let us recall the notion of condition (SSP).

Definition 2.2. Let A be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$. We say that A satisfies *condition* (SSP), if for any sequence of points $\{a_m\}$ of $\mathbb{R}^n$ tending to $0 \in \mathbb{R}^n$, such that $\lim_{m\to\infty} \frac{a_m}{\|a_m\|} \in D(A)$, there is a sequence of points $\{b_m\} \subset A$ such that,

$$\|a_m - b_m\| \ll \|a_m\|,\ \|b_m\|,$$

i.e. $\lim_{m\to\infty} \frac{\|a_m - b_m\|}{\|a_m\|} = 0$.

Below we give several general examples of sets satisfying condition (SSP), to illustrate the richness of this class. Consult [7] for more concrete and general examples.

Example 2.3. (1) Let $a_m := \frac{1}{m} \in \mathbb{R}$, $m \in \mathbb{N}$, and set $A := \{a_m\} \subset \mathbb{R}$. Then $0 \in \overline{A}$ and A satisfies condition (SSP).

Let $A \subseteq \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, then the following hold:

(2) The cone $LD(A)$ satisfies condition (SSP),
(3) If A is subanalytic or definable in some o-minimal structure, then it satisfies condition (SSP). See [4] for the definition of subanalytic, and see [2, 3] for the definitions of definable and o-minimal.
(4) If A is a finite union of sets, all of which satisfy condition (SSP), then A satisfies condition (SSP).
(5) If A is a C^1 manifold such that $0 \in A$, then it satisfies condition (SSP) and $LD(A) = T_0(A)$ i.e. the tangent space of A at $0 \in \mathbb{R}^n$ (this is not necessarily true for C^0 manifolds or if $0 \notin A$).
(6) Let $\pi : \mathcal{M}_n \to \mathbb{R}^n$ be the blowing-up at $0 \in \mathbb{R}^n$. It is not difficult to produce an example B which satisfies condition (SSP) and $\pi(B) = A$ does not necessarily satisfy (SSP). For instance we can take $B = C \cup E, E = \pi^{-1}(0), C \cap E = \{a\}$, such that C does not satisfy (SSP) and $LD(C) \subset LD(E)$ at a. Then $\pi(B) = \pi(C)$ does not satisfy (SSP), whereas B does satisfy (SSP).

(7) Let us denote by ℓ the positive x-axis, and by m the half line defined by $y = cx$, $x \geq 0$, for some $c > 0$. There are many types of zigzag curves B having infinitely many oscillations around $0 \in \mathbb{R}^2$ between ℓ and m. Some of them do not satisfy condition (SSP) e.g. Example 3.4 in [5], where the union of B and ℓ consists of similar triangles. See the example below 2.11 for more on condition (SSP) and zigzags.

Let $\pi : \mathcal{M}_2 \to \mathbb{R}^2$ be the blowing-up at $0 \in \mathbb{R}^2$. Using a local coordinate of $\mathcal{M}_2$, π is expressed by $\pi(X, Y) = (XY, Y)$. Let B be as above, with or without (SSP). Then we can see that $A := \pi(B)$ is in the region $|x| \leq c|y|^2$, $x \geq 0$, $y \geq 0$. Therefore $LD(A)$ is the positive y-axis. So regardless whether B has (SSP) or not, one can see that its image $A = \pi(B)$ satisfies condition (SSP). Compare to (6).

(8) Let $0 \in \overline{A} \cap \overline{B}$ and assume that $LD(A) \cap LD(B) = \{0\}$. Then $A \cup B$ has (SSP) if and only if both A and B have (SSP).

(9) A polynomially bounded strictly decreasing sequence $A = \{a_n \in \mathbb{R} | a_n > a_{n+1}, a_n \to 0, n \to \infty\}$ has (SSP). Note that there are examples of sequences, for instance $A = \{a_n = n^{-\sqrt{n}} \in \mathbb{R} | a_n > a_{n+1}, a_n \to 0, n \to \infty\}$ which are not polynomially bounded but they are (SSP).

Sketch:

We put $a_n = n^{-\alpha(n)}$ and we may assume that for all $n \geq 2$ we have $k - 1 < \alpha(n) \leq k$, for some integer k for which we have $n^{1-k} > a_n \geq n^{-k}$. By construction we have $(1 - \frac{1}{n+1})^{-\alpha(n)} > (n+1)^{\alpha(n) - \alpha(n+1)}$ and $(1 + \frac{1}{n})^{\alpha(n+1)} > n^{\alpha(n) - \alpha(n+1)}$ which in turn imply that $n^{\alpha(n) - \alpha(n+1)} \to 1, n \to \infty$ and this is equivalent to $\frac{a_n}{a_{n+1}} \to 1$ i.e. A has (SSP).

We have the following criterion for condition (SSP).

Proposition 2.4. *A satisfies condition (SSP) if and only if dist$(ta, A) = 0(t)$, for any direction $a \in DA$.*

Remark 2.5. We can always construct a global Lipschitz extension of a given Lipschitz mapping $f : A \to \mathbb{R}^n, A \subset (X, d)$ to $\tilde{f} : X \to \mathbb{R}^n$. Indeed, for a Lipschitz function with constant L, $f : A \to \mathbb{R}$, $A \subset X$, A endowed with the induced metric from (X, d), we have an extension formula (see H. Whitney [10] or S. Banach [1]):

$$\alpha(x) := \inf_{a \in A}(f(a) + Ld(x, a)).$$

Similarly one can extend it by

$$\beta(x) := \sup_{a \in A}(f(a) - Ld(x, a)).$$

This construction can be used to extend Lipschitz maps as well, however, without preserving the Lipschitz constant.

Remark 2.6. For a given Lipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ we can associate the bi-Lipschitz mappings $Y_+(f) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined by $Y_+(f)(x, y) := (x, y + f(x))$ and $Y_-(f) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined by $Y_-(f)(x, y) := (x + f(y), y)$.

Given a bi-Lipschitz mapping $\phi : A \to B, A, B \subset \mathbb{R}^n$, we can extend both ϕ and ϕ^{-1} to $\mathbb{R}^n$, say to global Lipschitz mappings $\tilde{\phi}$ and $\tilde{\phi}^{-1}$, and then consider the corresponding bi-Lipschitz mappings Y_+, Y_-.

We then consider the globally defined bi-Lipschitz $\tilde{\tilde{\phi}} := Y_-(\tilde{\phi}^{-1})^{-1} \circ Y_+(\tilde{\phi})$ and note that $\tilde{\tilde{\phi}}(x, 0) = (0, \phi(x)), \forall x \in A$.

In other words, in considering the direction cones of A and B we may assume that $\phi : A \to B$, $A, B \subset \mathbb{R}^n$, is defined globally, see [9]. This is a standard way of creating bi-Lipschitz mappings, we call it the doubling process.

Remark 2.7. Given a bi-Lipschitz homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$, one can consider $\psi_n(x) = n\psi(\frac{x}{n})$ and observe that in a compact neighbourhood of the origin we can, via Arzela-Ascoli's theorem, claim the existence of a limit $\psi_{n_i} \to d\psi$ (see [9]). This will be bi-Lipschitz as well with the same constants.

We have the following lemma in the (SSP) setting.

Lemma 2.8. *Let A, $B \subset \mathbb{R}^n$ set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $\phi : A \to B$ be a bi-Lipschitz homeomorphism. If A satisfies condition (SSP), then $d\tilde{\tilde{\phi}}(LD(A)) \subset LD(B)$.*

Proof. By remark 2.6 we may assume that ϕ is global and by remark 2.7 we can consider the associated $d\phi = \lim_{i\to\infty} \phi_{n_i}$. Take an arbitrary $v \in LD(A)$. Since A satisfies condition (SSP), there is a sequence of points $v_i \in A$, $i \in \mathbb{N}$, such that

$$\|v_i - \frac{v}{n_i}\| << \frac{1}{n_i} \approx \|v_i\|.$$

Accordingly we have

$$\|\phi(v_i) - \phi(\frac{v}{n_i})\| << \frac{1}{n_i},$$

which in turn shows that

$$\|n_i\phi(v_i) - n_i\phi(\frac{v}{n_i})\| \to 0 \ \text{ as } \ i \to \infty.$$

It follows that $d\phi(v) \in LD(B)$ as claimed (see [9]). □

Remark 2.9. In fact one can prove the following. Let A, $B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $\phi : (\mathbb{R}^n, A, 0) \to (\mathbb{R}^n, B, 0)$ be a Lipschitz mapping-germ. If A satisfies condition (SSP), then $d\phi(LD(A)) \subset LD(B)$. Here $d\phi$ is merely Lipschitz.

As a corollary of the above lemma, we have the generalised result of Theorem 3.2 in [9] to the case in the (SSP) setting.

Theorem 2.10. *Let A, $B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $\phi : A \to B$ be a bi-Lipschitz homeomorphism. If both A, B satisfy condition (SSP), then $d\tilde{\tilde{\phi}}(LD(A)) = LD(B)$.*

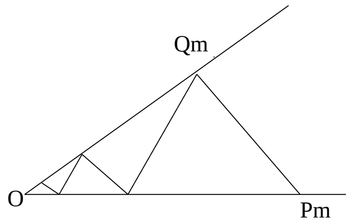

FIGURE 1

Example 2.11. Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a continuous zigzag function like in Figure 1, whose graph has infinitely many oscillations around $0 \in \mathbb{R}^2$ between the positive x-axis ℓ and the half line m defined by $y = cx$, $x \geq 0$, for some $c > 0$.

Now we define the mapping $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by

$$\phi(x, y) = Y_+(f) = (x, y + f(x)).$$

Then ϕ is a homeomorphism. Let us remark that ℓ satisfies condition (SSP), $LD(\ell) = \ell$ and $LD(\phi(\ell))$ is the sector surrounded by ℓ and m with $0 \in \mathbb{R}^2$ as the vertex. Therefore, by Theorem 2.10, we can see the following property:

If the zigzag $\phi(\ell)$ satisfies condition (SSP), then ϕ cannot be bi-Lipschitz (i.e. f cannot be Lipschitz). In other words, if ϕ is a bi-Lipschitz homeomorphism (f is Lipschitz), then the zigzag $\phi(\ell)$ does not satisfy condition (SSP).

The above property follows also from some directional property of intersection set (Proposition 2.29 and Appendix in [7]) or an important property concerning $LD(h(A)) = LD(h(LD(A)))$ in [5].

Using Theorem 2.10 , we can show the following corollaries.

Corollary 2.12. *Let A be a set germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A \setminus \{0\}}$, and let $0 \in \mathbb{R}^n$ have a neighbourhood in A bi-Lipschitz homeomorphic to an open set in some Euclidean space $\mathbb{R}^k$. Then $LD(A)$ is bi-Lipschitz homeomorphic to $\mathbb{R}^k$.*

Proof. Assume that A is bi-Lipschitz homeomorphic to an open set $U \subset \mathbb{R}^k$. Then according to example 2.3 (5), U satisfies condition (SSP) and $LD(U) = \mathbb{R}^k$. Therefore by Theorem 2.10, their tangent cones are bi-Lipschitz homeomorphic as well. $\square$

Corollary 2.13. *Let A be a set germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A \setminus \{0\}}$, and let $0 \in \mathbb{R}^n$ have a neighbourhood V in A bi-Lipschitz homeomorphic to a cone $LD(C)$. Then V and $LD(A)$ are bi-Lipschitz homeomorphic as well, in particular $dimD(A) = dimA - 1$.*

Proof. Any cone has (SSP) and $LD(LD(C)) = LD(C)$. Therefore by Theorem 2.10, their tangent cones are bi-Lipschitz homeomorphic as well. $\square$

We can show also the following lemma in the (SSP) setting.

Lemma 2.14. *Let $A,\ B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz homeomorphism. If both A, B satisfy condition (SSP), then*

$$\dim(D(h(A)) \cap D(h(B))) \geq \dim(D(A) \cap D(B)).$$

Proof. Having established Lemma 2.8, the proof follows as in [9]. □

As a consequence of the above lemma, we have the generalised result of Main Theorem in [5] to the case in the (SSP) setting.

Theorem 2.15. *Let $A,\ B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $A,\ B,\ h(A),\ h(B)$ satisfy condition (SSP). Then we have the equality of dimensions,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

Definition 2.16. Let $A,\ B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. We say that A, B are *transverse* at $0 \in \mathbb{R}^n$ if and only if:

$$\dim LD(A) + \dim \mathrm{LD(B)} - \dim(LD(A) \cap LD(B)) = n.$$

As a corollary of Theorem 2.15, we have the following preserving of transversality result.

Corollary 2.17. *Let $A,\ B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $A,\ B,\ h(A),\ h(B)$ satisfy condition (SSP). Then A and B are transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are transverse at $h(0) = 0 \in \mathbb{R}^n$.*

On the other hand in [7] we introduced a notion of weak transversality and showed in Theorem 3.5 that weak transversality is preserved under rather mild assumptions. We are going to recall the result for reader convenience.

Definition 2.18. Let $A,\ B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$. We say that A, B are *weakly transverse* at $0 \in \mathbb{R}^n$ if and only if $D(A) \cap D(B) = \emptyset$.

Theorem 2.19. *Let $A,\ B$ be two set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A or B satisfies condition (SSP), and $h(A)$ or $h(B)$ satisfies condition (SSP). Then A and B are weakly transverse at $0 \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $0 \in \mathbb{R}^n$.*

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