Homology ring mod 2 of Free Loop Groups of Spinor Groups *

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Abstract

In this paper the induced homology map of spinor group on its base-pointed loop space is computed and we deduce the Hopf algebra structure of the homology ring of the free loop group of $Spin(N)$.

1 Introduction

Let G, ΩG be a compact connected Lie group and its based loop space, whose multiplication maps are μ, λ respectively. We denote the free loop group of G by $LG(G)$, which is the topological group of all continuous maps from S^1 to G. In [8] the cohomology of $LG(G)$ is well investigated. $LG(G)$ contains ΩG as its normal subgroup and $LG(G)$ is the semi-direct product of ΩG and G. Thus $LG(G)$ is homeomorphic to $\Omega G \times G$ and we have the following commutative diagram:

$$\begin{array}{ccccc}
\Omega G \times G \times \Omega G \times G & \xrightarrow{\omega} & \Omega G \times \Omega G \times G \times G & \xrightarrow{\lambda \times \mu} & \Omega G \times G \\
\downarrow{\scriptstyle \Phi \times \Phi} & & & & \downarrow{\scriptstyle \Phi} \\
LG\,(G) \times LG\,(G) & & \xrightarrow{\qquad\qquad \lambda' \qquad\qquad} & & LG\,(G)
\end{array}$$

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where $\Phi : \Omega G \times G \to LG\ (G)$ is a map defined by $\Phi(l,g)(t) = l(t)\cdot g$, λ' is the multiplication map of $LG(G)$ and ω is the composition

$$(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G} \times 1_G \times \mathrm{Ad} \times 1_G) \circ (1_{\Omega G} \times \Delta_G \times 1_{\Omega G \times G}).$$

Here Ad is the point-wise adjoint action of G on ΩG and T is the transposition. This shows that the knowledge of the Hopf algebra structures of the homology rings of G, ΩG and the induced homology map of Ad enable us to know the Hopf algebra structure of the homology ring of $LG(G)$.

Moreover, in [8], it is showed that provided G is simply connected, $H_*(Ad; Z/p)$ is equal to the induced homology map of second projection if and only if $H^*(G;Z)$ is p-torsion free. This implies that, if we assume G is simple, non-trivial cases are $G = E_8$ for $p = 5$, $G = F_4, E_6, E_7, E_8$ for $p = 3$ and $G = G_2, F_4, E_6, E_7, E_8, Spin(N)$ for $p = 2$. The induced homology map of Ad in all these cases except for $(G,p) = (Spin(N), 2)$ were investigated in Hamanaka [2], [3], Hamanaka & Hara [4] and Hamanaka & Hara & Kono [5].

In this paper we compute Ad_*, the induced homology map of Ad for the case of $(G,p) = (Spin(N), p)$ and consider the mod 2 homology ring of $LG\ (Spin(N))$. In the rest of this paper we use $\mathbf{Z}/2\mathbf{Z}$ as the coefficient ring of cohomology and homology unless mentioned.

2 Generating variety

$Spin(N)$ is the universal covering group of $SO(N)$ and the mod 2 cohomology of them are well known as follows. See [7].

Theorem 2.1. *We denote the covering map $Spin(N) \to SO(N)$ by p. Then*

$$\begin{aligned} \mathrm{H}^*\,(SO(N)) &= \Delta(x_1, x_2, x_3, \cdots x_{N-1}), \\ \mathrm{H}^*\,(Spin(N)) &= \Delta(\{x_\alpha | \alpha \in A\}) \otimes \bigwedge u \end{aligned}$$

where

$$A = \{1, 2, \cdots, N-1\} \backslash \{\textit{powers of 2}\},$$

$$|x_i| = i,\ |u| = 2^k - 1,\ 2^{k-1} < N \le 2^k,$$

x_i's are primitive and the coproduct of u is

$$u \otimes 1 + 1 \otimes u + \sum_{\substack{i,j \neq 2^l,\, i<j \\ i+j=2^k-1}} x_i \otimes x_j.$$

Also $p^*(x_\alpha) = x_\alpha$ for $\alpha \in A$.

Here we have the following diagram:

$$\begin{array}{ccc} Spin(N) \times \Omega Spin(N) & \stackrel{\mathrm{Ad}}{\longrightarrow} & \Omega Spin(N) \\ \downarrow p \times \Omega p & & \| \\ SO(N) \times \Omega_0 SO(N) & \stackrel{\mathrm{Ad}_0}{\longrightarrow} & \Omega_0 SO(N) \end{array}$$

where $\Omega_0 SO(N)$ is the connected component of $\Omega SO(N)$ containing null homotopic loops and Ad_0 is the pointwise adjoint action of $SO(N)$ on $\Omega_0 SO(N)$.

This diagram and the above theorem tells that

$$\mathrm{Ad}_* = \mathrm{Ad}_{0*} \circ p_* \tag{1}$$

and what we have to compute is Ad_{0*}.

From now on, to set the generators of $\mathrm{H}_*(\Omega_0 SO(N))$ we remember the R.Bott's theorem of generating variety. (See R.Bott [1].)

Let G be a compact connected Lie group and $s : S^1 \to G$ be a injective homomorphism. We denote the centralizer of s in G by G_s and the quotient space G/G_s by G^s. Then there is a map $g^s : G^s \to \Omega_0 G$ defined by

$$g^s([x])(t) = [x, s(t)] = xs(t)x^{-1}s(t)^{-1}$$

where $x \in G$, $t \in S^1$, and $[x]$ is the class represented by x. Moreover, let T be a maximal torus containing s and $W(G,T)$ be the Weyl group of G. Let $\boldsymbol{s} \in \mathrm{H}_1(T;\mathbf{Z})$ be the homology class determined by s and Λ_s be the submodule of $\mathrm{H}_*(T;\mathbf{Z})$ generated by $W(G,T)\boldsymbol{s}$.

We call $s \in T \subset G$ a generating circle for G if each root $\theta \in \mathrm{H}^1(T;\mathbf{Z})$ takes on the value 1 on some element of $\bigwedge_s$. Also we call G^s a generating variety and g^s a generating map.

Theorem 2.2. *(R.Bott) If s is a generating circle for G, then the image of $g^s_* : \mathrm{H}_*(G^s;\mathbf{Z}) \to \mathrm{H}_*(\Omega_0 G;\mathbf{Z})$ generates the Pontrjagin ring $\mathrm{H}_*(\Omega_0 G;\mathbf{Z})$.*

R.Bott applied this theorem to the case of $G = SO(N)$. It is known that $s : SO(2) \hookrightarrow SO(N)$ is a generating circle and the mod 2 cohomology ring of generating variety $G^s = SO(N)/(SO(2) \times SO(N-2))$ is as follows. See Ishitoya [6] for detail.

- If $N = 2m+2$,

$$\mathrm{H}^*\left(\frac{SO(N)}{SO(2)\times SO(N-2)}\right) = \mathbf{Z}/2\mathbf{Z}[t,s]/(t^{m+1}, s^2 - \delta_m t^m s)$$

  where $|t| = 2$, $|s| = 2m$ and

$$\delta_m = \begin{cases} 0 & m \text{ is odd,} \\ 1 & m \text{ is even.} \end{cases}$$

- If $N = 2m+1$,

$$\mathrm{H}^*\left(\frac{SO(N)}{SO(2)\times SO(N-2)}\right) = \mathbf{Z}/2\mathbf{Z}[t,s]/(t^m, s^2)$$

  where $|t| = 2$, $|s| = 2m$.

We take the dual element of t^i, st^i with respect to the monomial basis and denote them by $b_i, b'_i \in \mathrm{H}_*(G^s)$ respectively. (We put $b_0 = 1$.) Then the generators of $\mathrm{H}_*(G^s)$ as module are

$$\begin{array}{ll} \{b_1, b_2, \cdots, b_m, b'_m, b'_{m+1}, \cdots b'_{2m}\} & \text{if } N = 2m+2, \\ \{b_1, b_2, \cdots, b_{m-1}, b'_m, b'_{m+1}, \cdots b'_{2m-1}\} & \text{if } N = 2m+1. \end{array}$$

We also denote $g^s_* b_i, g^s_* b'_i \in \mathrm{H}_*(\Omega_0 SO(N))$ by b_i, b'_i and regard $\mathrm{H}_*(G^s)$ as a submodule of $\mathrm{H}_*(\Omega SO(N))$ by g^s_*, because g^s_* is injective. Here by Theorem 2.2 b_i, b'_i's are generators of $\mathrm{H}_*(\Omega_0 SO(N))$ and R.Bott shows their relations:

Theorem 2.3. *(R.Bott)*

1. *If $N = 2m+2$, $\mathrm{H}_*(\Omega_0 SO(N))$ is generated by b_i, b'_{m+i} ($i = 0, 1, \cdots, m$) and the relations are*

$$\begin{cases} {b_j}^2 = 0 & j < \frac{m}{2} \\ {b_j}^2 = \sum_{i=0}^{2j-m} (b_i b'_{2j-i}) & j \geq \frac{m}{2}. \end{cases}$$

2. *If $N = 2m+1$, $\mathrm{H}_*(\Omega_0 SO(N))$ is generated by b_i, b'_{m+i} ($i = 0, 1, \cdots, m-1$) and the relations are the same as the case of 1.*

In other words, by the induced homology map of the natural inclusion $\Omega_0 SO(2m+1) \to \Omega_0 SO(2m+2)$*,* $\mathrm{H}_*(\Omega_0 SO(2m+1))$ *is a subring of* $\mathrm{H}_*(\Omega_0 SO(2m+2))$*.*

Remark In [1], all the computations are done with its coefficient ring Z and notations are different from those in this paper. If $N = 2n+1$, b_j, b'_{n+j} $(0 \le j \le n-1)$ in this paper are mod 2 reduction of σ_j, $2\sigma_{n+j}$ in [1] respectively and if $N = 2n+2$, b_j, $(1 \le j \le m-1)$, $b_{n+j'}$ $(1 \le j' \le m)$, b_n and b'_n in this paper are mod 2 reduction of σ_j, $2\sigma_{n+j'}$, $\sigma_n + \epsilon$ and 2ϵ in [1] respectively. Here remark that σ_j's and ϵ are elements of $\mathrm{H}_*(\Omega_0 SO(N); \mathbf{Q})$.

Moreover the coproduct can be determined easily since we know the algebra structure of the cohomology ring of the generating variety.

Theorem 2.4. *(R.Bott) In Theorem 2.3 the coproduct is given by*

$$\begin{aligned}
\Delta_* b_j &= \sum_{\alpha+\beta=j} b_\alpha \otimes b_\beta \qquad (1 \le j \le m) \\
\Delta_* b'_{m+j} &= \sum_{\alpha+\beta=j} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{\beta+m}) \qquad (0 \le j \le m-1) \\
\Delta_* b'_{2m} &= \sum_{\alpha+\beta=m} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{\beta+m}) + \delta_m (b'_m \otimes b'_m)
\end{aligned}$$

From now on we consider the even case, because, by the above theorem, one can deduce the result for odd case from that for even case.

3 Action of $SO(N-2)$ on $\Omega_0 SO(N)$

In this section we set $N = 2m+2$ and $G = SO(2m+2)$. We consider a map $\alpha : SO(2m+2) \times G^s \to G^s$ defined by

$$\alpha(g, [x]) = [gx] \quad \text{for } g \in SO(2m+2) \text{ and } [x] \in G^s \ .$$

Lemma 3.1. *The following diagram commutes.*

$$\begin{array}{ccc}
SO(2m) \times G^s & \xrightarrow{\alpha|_{SO(2m)\times G^s}} & G^s \\
\downarrow 1 \times g^s & & \downarrow g^s \\
SO(2m) \times \Omega_0 G & \xrightarrow{Ad_0} & \Omega_0 G
\end{array}$$

Proof. By the definition, for $g \in SO(2m)$, $x \in SO(2m)$ and $t \in S^1$,

$$\begin{aligned} g^s(\alpha(g,[x]))(t) &= g^s([gx])(t) \\ &= gxs(t)x^{-1}g^{-1}s(t)^{-1} \\ &= gxs(t)x^{-1}s(t)^{-1}g^{-1} \\ &= \mathrm{Ad}_0(g, g^s(x))(t). \end{aligned}$$

∎

Now we denote $\mathrm{Ad}_{0*}(y \otimes b)$ by $y * b$ for $y \in \mathrm{H}_*(SO(N))$ and $b \in \mathrm{H}_*(\Omega_0 SO(N))$. Then above diagram says:

Theorem 3.2. *If $y \in \mathrm{H}_*(SO(2m)) \subset \mathrm{H}_*(SO(2m+2))$ then*

$$y * b = \alpha_*(y \otimes b)$$

for $b \in \mathrm{H}_*(G^s) \subset \mathrm{H}_*(\Omega_0 G)$.

To compute α_* we start with the computation in cohomology.

Lemma 3.3.

$$\alpha^*(t) = 1 \otimes t$$

Proof. We set $H = SO(2m) \times SO(2)$. In the following diagram $p_2 : SO(2m+2) \times (EG \times_G G/H) \to EG \times_G G/H$ and $p_2' : EG \times G/H \to G/H$ are second projections, $q : EG \times G/H \to EG \times_G G/H$ is the natural projection and $\beta : SO(2m+2) \times EG \times G/H \to EG \times G/H$ is defined as

$$\beta(h, g, x) = (gh^{-1}, hx)$$

for $h \in G, g \in EG$ and $x \in G/H$.

$$\begin{array}{ccccc} SO(2m+2) \times G/H & \xrightarrow{\alpha} & G/H & & \\ \uparrow 1_G \times p_2' & & \uparrow p_2' & & \\ SO(2m+2) \times EG \times G/H & \xrightarrow{\beta} & EG \times G/H & & \\ \downarrow 1_G \times q & & \downarrow q & & \\ SO(2m+2) \times (EG \underset{G}{\times} G/H) & \xrightarrow{p_2} & EG \underset{G}{\times} G/H & \cong & BH \end{array}$$

Then this diagram commutes. Here remark that $EG \times_G G/H = BH$ and p_2' is a homotopy equivalence. Thus if we set $\phi = q \circ {p_2'}^{-1}$, ϕ is a classifying map $G/H \to BH$ of the fibration $H \to G \to G/H$.

Also, since t is in the transgression image with regard to the fibration $H \to G \to G/H$, t is in the image of ϕ^*, that is, there exists $t' \in \mathrm{H}^*(BH)$ such that $\phi^*(t') = t$. Thus

$$\begin{aligned}
\alpha^*(t) &= \alpha^* \phi^*(t') \\
&= (1_G \times {p_2'}^{-1})^* \circ (1_G \times q)^* \circ p_2^*(t') \\
&= 1 \otimes \phi^*(t') \\
&= 1 \otimes t
\end{aligned}$$

∎

To compute $\alpha^*(s)$, we consider the commutative diagram below:

$$\begin{array}{ccc}
SO(2m+2) \times \dfrac{SO(2m+2)}{SO(2){\times}SO(2m)} & \xrightarrow{\alpha} & \dfrac{SO(2m+2)}{SO(2){\times}SO(2m)} \\
\uparrow 1 \times p' & & \uparrow p' \\
SO(2m+2) \times \dfrac{SO(2m+2)}{U(1){\times}U(m)} & \xrightarrow{\alpha'} & \dfrac{SO(2m+2)}{U(1){\times}U(m)} \\
\downarrow 1 \times p'' & & \downarrow p'' \\
SO(2m+2) \times \dfrac{SO(2m+2)}{U(m+1)} & \xrightarrow{\alpha''} & \dfrac{SO(2m+2)}{U(m+1)} \\
\uparrow 1 \times p''' & & \uparrow p''' \\
SO(2m+2) \times SO(2m+2) & \xrightarrow{\mathrm{ad}} & SO(2m+2)
\end{array}$$

Here α', α'' are actions of $SO(2m+2)$ by the left translation and ad is the usual adjoint action and p', p'', p''' are natural projections induced by standard embeddings of subgroups. The cohomology rings of spaces in the diagram are shown as follows. See [6] for detail.

$$\begin{aligned}
\mathrm{H}^*\left(\frac{SO(2m+2)}{U(1){\times}U(m)}\right) &\cong \mathrm{H}^*\left(\frac{SO(2m+2)}{U(m+1)}\right)[t]/(t^{m+1}), \\
\mathrm{H}^*\left(\frac{SO(2m+2)}{U(m+1)}\right) &\cong \Delta(x_2, x_4, \cdots x_{2m}).
\end{aligned}$$

Here by the induced cohomology map of the natural projection $SO(2m+2) \to SO(2m+2)/U(m+1)$, $x_{2i} \in \mathrm{H}^*(SO(2m+2)/U(m+1))$ is mapped to $x_{2i} \in \mathrm{H}^*(SO(2m+2))$ and also, in [6] it was shown that

$$\begin{aligned} p'^*(s) &= \delta_m t^m + e, \\ p'^*(t) &= t \end{aligned}$$

where $e = \sum_{i=1}^m (-1)^i x_{2i} t^{m-i}$. This implies p'^* is injective.

We can easily show that

$$\mathrm{ad}^*(x_{2i}) = 1 \otimes x_{2i} + x_{2i} \otimes 1$$

and this implies

$$\alpha''^*(x_{2i}) = 1 \otimes x_{2i} + x_{2i} \otimes 1$$

and

$$\alpha'^*(x_{2i}) = 1 \otimes x_{2i} + x_{2i} \otimes 1.$$

Thus we can proceed the computation as:

$$\begin{aligned} (1 \times p')^* \circ \alpha^*(s) &= \alpha'^* \circ p'^*(s) \\ &= \alpha'^*(\delta_m t^m + e) \\ &= \delta_m(1 \otimes t^m) + 1 \otimes e + \sum_{i=1}^m x_{2i} \otimes t^{m-i} \\ &= (1 \times p')^*(1 \otimes s + \sum_{i=1}^m x_{2i} \otimes t^{m-i}). \end{aligned}$$

By the injectivity of p'^*, we have :

Lemma 3.4.

$$\alpha^*(s) = 1 \otimes s + \sum_{i=1}^m e_{2i} \otimes t^{m-i}.$$

Now, we dualize the results of Lemma 3.3 and Lemma 3.4. We denote the dual element of x_i with regard to the monomial basis by y_i. Then we obtain

$$\begin{aligned} \alpha_*(y_{\mathrm{odd}} \otimes b) &= 0 \quad \text{for any } b \in \mathrm{H}_*(G/H), \\ \alpha_*(y_{2i} \otimes b_j) &= \begin{cases} 0 & i+j < m, \\ b_i & i+j \geq m, \end{cases} \\ \alpha_*(y_{2i} \otimes b'_{m+j}) &= 0 \end{aligned}$$

for $1 \leq i \leq m$ and $0 \leq j \leq m$. By Theorem 3.2 it immediately follows that:

Proposition 3.5. *For $y_{2i} \in \mathrm{H}_*(SO(2m+2))$ ($i \leq m-1$), $b_j, b'_{m+j} \in \mathrm{H}_*(\Omega_0 SO(2m+2))$ ($0 \leq j \leq m$) and any $b \in \mathrm{H}_*(\Omega_0 SO(2m+2))$,*

$$\begin{aligned} y_{\mathrm{odd}} * b &= 0, \\ y_{2i} * b_j &= \begin{cases} 0 & i+j < m, \\ b_i & i+j \geq m, \end{cases} \\ y_{2i} * b'_{m+j} &= 0. \end{aligned}$$

4 Action of $SO(N)$ on $\Omega_0 SO(N)$

Concerning the computation of Ad_{0*}, all we have to do is to compute $y_{2m} * b_j$ and $y_{2m} * b'_{m+j}$ for $0 \leq j \leq m$. To do this we use the diagram below:

$$\begin{array}{ccc} SO(2m+2) \times \Omega_0 SO(2m+2) & \overset{\mathrm{Ad}_{2m+2}}{\longrightarrow} & \Omega_0 SO(2m+2) \\ i \times \Omega_0 i \downarrow & & \downarrow \Omega_0 i \\ SO(2m+4) \times \Omega_0 SO(2m+4) & \overset{\mathrm{Ad}_{2m+4}}{\longrightarrow} & \Omega_0 SO(2m+4) \end{array}$$

To avoid confusion, we denote $\mathrm{Ad} : SO(n) \times \Omega_0 SO(n) \to \Omega_0 SO(n)$ by Ad_n. By this diagram and Proposition 3.5, we obtain

$$\left.\begin{aligned} \Omega_0 i_*(y_{2m} * b_j) &= \mathrm{Ad}_{2m+4*} \circ (i \times \Omega_0 i)_*(y_{2m} \otimes b_j) = b'_{m+j}, \\ \Omega_0 i_*(y_{2m} * b'_{m+j}) &= \mathrm{Ad}_{2m+4*} \circ (i \times \Omega_0 i)_*(y_{2m} \otimes b'_{m+j}) = 0. \end{aligned}\right\} \tag{2}$$

These tell that $y_{2m} * b_j$, $y_{2m} * b'_j$ was determined up to $\mathrm{Ker}(\Omega_0 i_*)$.

Lemma 4.1. $\mathrm{Ker}(\Omega_0 i_*) = (\mathrm{b}'_{\mathrm{m}})$.

Proof. This can be checked by the fact

$$\left.\begin{aligned} \Omega_0 i_* b_j &= b_j \ (\ 1 \leq j \leq m\ ), \\ \Omega_0 i_* b'_m &= 0, \\ \Omega_0 i_* b'_j &= b'_j \ (\ m+1 \leq j \leq 2m\ ) \end{aligned}\right\} \tag{3}$$

and the relations in Theorem 2.3.

Also (3) can be obtained by dualizing the induced cohomology map $SO(2m+2)/SO(2) \times SO(2m) \to SO(2m+4)/SO(2) \times SO(2m+2)$ since the generating maps commute with the respective inclusion maps. ∎

By (2) and the above lemma, we can say

$$\begin{aligned} y_{2m} * b_j &= b'_{m+j} + b'_m c_j, \\ y_{2m} * b'_j &= b'_m c'_{m+j} \end{aligned}$$

for some $c_j \in \mathrm{H}_j(\Omega_0 SO(2m+2))$, $c'_{m+j} \in \mathrm{H}_{m+j}(\Omega_0 SO(2m+2))$. To determine c_j and c'_{m+j} we use some properties of 'adjoint action on homology' stated in the following theorem. See [2] and Kono & Kozima [8] for its proof.

Theorem 4.2. *For y, y', $y'' \in \mathrm{H}_*(G)$ and b, $b' \in \mathrm{H}_*(\Omega G)$*

(i) $1 * b = b$ *and* $y * 1 = 0$, *if* $|y| > 0$.

(ii) $\Delta_*(y * b) = \sum (y' * b') \otimes (y'' * b'')$ *where* $\Delta_* y = \sum y' \otimes y''$ *and* $\Delta_* b = \sum b' \otimes b''$. *And also* $\overline{\Delta}_*(y * b) = (\Delta_* y) * (\overline{\Delta}_* b)$.

(iii) $(yy') * b = y * (y' * b)$.

(iv) $y * (bb') = \sum (y' * b)(y'' * b')$ *where* $\Delta_* y = \sum y' \otimes y''$.

(v) *If b is primitive then $y * b$ is primitive.*

Proposition 4.3.

$$\begin{array}{rclr} y_{2m} * b_j &=& b'_{m+j} + b'_m b_j & j \le m-1 \\ y_{2m} * b_m &=& b'_{2m} + b'_m b_m + \rho {b'_m}^2 & j = m \end{array} \tag{4}$$

for some $\rho \in \mathbf{Z}/2\mathbf{Z}$.

Proof. By Theorem 2.4 and Theorem 4.2 (ii), we can compute the coproduct of $y_{2m} * b_j$ for $1 \le j \le m$ as:

$$\begin{aligned} \Delta_*(y_{2m} * b_j) &= \sum_{\alpha+\beta=j} (y_{2m} * b_\alpha) \otimes b_\beta + \sum_{\alpha+\beta=j} (y_m * b_\alpha) \otimes (y_m * b_\beta) \\ &\quad + \sum_{\alpha+\beta=j} b_\alpha \otimes (y_{2m} * b_\beta). \end{aligned} \tag{5}$$

Here remark that by Proposition 3.5 $y_m * b_\alpha \neq 0$ if and only if $\alpha \ge \frac{m}{2}$. This implies $\sum_{\alpha+\beta=j} (y_m * b_\alpha) \otimes (y_m * b_\beta)$ vanishes if $j \neq m$.

Now we use induction. When $j = 1$, the above formula says

$$\overline{\Delta}_*(y_{2m} * b_1) = 0$$

where $\overline{\Delta}_*(w) = \Delta_*(w) - 1 \otimes w - w \otimes 1$. Thus $b'_{m+1} + b'_m c_1$ is primitive and $c_1 = b_1$.

For $1 < j \leq m-1$, if we assume that $c_k = b_k$ for $1 \leq k < j$, then

$$\begin{aligned}
\overline{\Delta}_*(y_{2m} * b_j) &= \sum_{\substack{\alpha+\beta=j \\ \alpha>0,\beta>0}} \left\{(b'_{m+\alpha} + b'_m b_\alpha) \otimes b_\beta + b_\alpha \otimes (b'_{m+\beta} + b'_m b_\beta)\right\} \\
&= \sum_{\substack{\alpha+\beta=j \\ \alpha>0,\beta>0}} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{m+\beta}) \\
&\qquad + (b'_m \otimes 1 + 1 \otimes b'_m) \sum_{\substack{\alpha+\beta=j \\ \alpha>0,\beta>0}} (b_\alpha \otimes b_\beta) \\
&= \overline{\Delta}_*(b'_{m+j} + b'_m b_j). \qquad (6)
\end{aligned}$$

Since there is no primitive element of the form $b'_m c_j$ in $\mathrm{H}_{m+j}\left(\Omega_0 SO(2m+2)\right)$, $c_j = b_j$.

Next we consider the case $j = m$ and assume $c_j = b_j$ for $1 \leq j \leq m-1$. If m is odd, since $\sum_{\alpha+\beta=j}(y_m * b_\alpha) \otimes (y_m * b_\beta) = 0$, (6) is true for $j = m$. Also if m is even,

$$\begin{aligned}
\overline{\Delta}_*(y_{2m} * b_m) &= \sum_{\substack{\alpha+\beta=m \\ \alpha>0,\beta>0}} \left\{(b'_{m+\alpha} + b'_m b_\alpha) \otimes b_\beta + b_\alpha \otimes (b'_{m+\beta} + b'_m b_\beta)\right\} \\
&\qquad + (y_m * b_{m/2}) \otimes (y_m * b_{m/2}) \\
&= \sum_{\substack{\alpha+\beta=m \\ \alpha>0,\beta>0}} (b'_{m+\alpha} \otimes b_\beta + b_\alpha \otimes b'_{m+\beta}) \\
&\qquad + (b'_m \otimes 1 + 1 \otimes b'_m) \sum_{\substack{\alpha+\beta=m \\ \alpha>0,\beta>0}} (b_\alpha \otimes b_\beta) + b'_m \otimes b'_m \\
&= \overline{\Delta}_*(b'_{2m} + b'_m b_m).
\end{aligned}$$

and (6) remains true for this case. Thus $y_{2m} * b_m$ is determined up to primitive elements:

$$y_{2m} * b_m = b'_{2m} + b'_m b_m + \rho {b'_m}^2 \quad \rho \in \mathbf{Z}/2\mathbf{Z}.$$

∎

Proposition 4.4.

$$\begin{aligned}
y_{2m} * b'_{m+j} &= b'_m b'_{m+j} & & 0 \leq j \leq m-1 \\
y_{2m} * b'_{2m} &= b'_m b'_{2m} + \rho {b'_m}^3 & & j = m
\end{aligned}$$

where ρ is the same as in Proposition 4.3.

Proof. By Theorem 4.2 (iii),(iv),

$$\begin{aligned} y_{2m} * b'_m &= y_{2m} * (y_2 * b_{m-1}) \\ &= y_2 * (y_{2m} * b_{m-1}) \\ &= y_2 * (b'_{2m-1} + b'_m b_{m-1}) \\ &= {b'_m}^2. \end{aligned}$$

For $1 \leq j \leq m-1$, in the similar way we have

$$\begin{aligned} y_{2m} * b'_{m+j} &= y_{2m} * (y_{2j} * b_m) \\ &= y_{2j} * (y_{2m} * b_m) \\ &= y_{2j} * (b'_{2m} + b'_m b_m + \rho {b'_m}^2) \\ &= b'_m b'_{m+j}. \end{aligned}$$

Finally, it follows in the same way that

$$\begin{aligned} y_{2m} * b'_{2m} &= y_{2m} * (y_{2m} * b_m + b'_m b_m + \rho {b'_m}^2) \\ &= (y_{2m} * b'_m) b_m + b'_m (y_{2m} * b_m) \\ &= {b'_m}^2 b_m + b'_m (b'_{2m} + b'_m b_m + \rho {b'_m}^2) \\ &= b'_m b'_{2m} + \rho {b'_m}^3. \end{aligned}$$

∎

Now we summarize the above results.

Theorem 4.5. *For the generators y_k of $\mathrm{H}_*(SO(2m+2))$, b_j, b'_{m+j} of*
$\mathrm{H}_*(\Omega SO(2m+2))$ *$(1 \leq k \leq 2m+1,\ 0 \leq j \leq m)$,*

$$\begin{array}{lrcl} \textit{if } k \textit{ is odd} & y_{odd} * b_j &=& 0, \\ & y_{odd} * b'_{m+j} &=& 0, \\ \textit{if } i < m & y_{2i} * b_j &=& \begin{cases} 0 & i+j < m, \\ b'_{i+j} & i+j \geq m, \end{cases} \\ & y_{2i} * b'_{m+j} &=& 0, \\ \textit{if } k = 2m \textit{ and } j \neq m & y_{2m} * b_j &=& b'_{m+j} + b'_m b_j, \\ & y_{2m} * b'_{m+j} &=& b'_m b'_{m+j}, \\ \textit{if } k = 2m \textit{ and } j = m & y_{2m} * b_m &=& b'_{2m} + b'_m b_m, \\ & y_{2m} * b_m &=& b'_m b'_{2m}. \end{array}$$

Above equations remain true for $y_k \in \mathrm{H}_*(SO(2m+1))$ and b_j, $b'_{m+j} \in \mathrm{H}_*(\Omega SO(2m+1))$. $(1 \leq k \leq 2m,\ 0 \leq j \leq m-1)$

Proof. We prove in §5 that ρ in Proposition 4.3 is zero. Then the statement for even m is true.

Because $\mathrm{H}_*(\Omega SO(2m+1))$ is a subalgebra of $\mathrm{H}_*(\Omega SO(2m+2))$ and adjoint actions commute with the respective inclusions, the whole statement follows. ∎

5 Determination of ρ in Proposition 4.3

To determine ρ we introduce a map η from $SO(2m+2) \otimes G^s$ to $\Omega_0 SO(2m+2)$ defined as follows.

Let f and g be the compositions of maps $\alpha \circ g^s$ and $(1 \times g^s) \circ \mathrm{Ad}_0$ respectively. We put η' as

$$\eta'(x,y) = f(x,y)g(x,y)^{-1}.$$

Refer to the diagram below.

$$\begin{tikzcd}
RP^{2m+1} \times G^s \arrow[r, "j_{2m+2} \times 1"] \arrow[d, "p_m \times 1"] & SO(2m+2) \times G^s \arrow[r, "\alpha"] \arrow[d, "1 \times g^s"] \arrow[dr, "\eta'"] & G^s \arrow[d, "g^s"] \\
RP^{2m+1}/RP^{2m-1} \times G^s \arrow[rr, bend right, "\eta"'] & SO(2m+2) \times \Omega_0 SO(2m+2) \arrow[r, "\mathrm{Ad}_0"] & \Omega_0 SO(2m+2)
\end{tikzcd}$$

Here remark that $\eta'|_{SO(2m) \times G^s}$ is the constant map by Lemma 3.1 and also $\eta' \circ \Omega_0 i$, where i is the inclusion of $SO(2m+2)$ into $SO(2m+4)$, is homotopic to the constant map.

There is an inclusion of RP^{n-1} into $SO(n)$ denoted by j_n, which is defined by attaching a line $l \in RP^{n-1}$ with the composition of a fixed orientation reversing self linear map A_n of $\mathbf{R}^n$ and the reflection in the direction of l. We set

$$A_n = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Since $\eta'|_{SO(2m)\times G^s} \simeq *$ and $j_{2m+2}|_{RP^{2m-1}} \subset SO(2m)$, there is a map $\eta : RP^{2m+1}/RP^{2m-1} \times G^s \to \Omega_0 SO(2m+2)$ where $\eta \circ (p_m \times 1) \simeq \eta' \circ (j_{2m+2} \times 1)$

First, we consider the next lemma.

Lemma 5.1. *In* $\mathbf{Z}/2\mathbf{Z}$ *homology,*

$$\eta'_*(y_{2m} \otimes b_m) = \rho {b'_m}^2.$$

Proof. By the definition, $\eta' = \lambda \circ (1 \times \iota) \circ (f \times g) \circ \Delta_{SO(2m+2)\times G^s}$ where ι is the map of taking the inverse in $\Omega_0 SO(2m+2)$. Thus we can proceed the calculation as:

$$\begin{aligned}
& \lambda_* \circ (1\times\iota)_* \circ (f\times g)_* \circ {\Delta_{SO(2m+2)\times G^s}}_*(y_{2m}\otimes b_m) \\
= \; & \lambda_* \circ (1\times\iota)_* \circ (f\times g)_* \{\sum_{j=0}^{m}(y_{2m}\otimes b_j)\otimes(1\otimes b_{m-j}) \\
& \qquad + \sum_{j=0}^{m}(y_m\otimes b_j)\otimes(y_m\otimes b_{m-j}) + \sum_{j=0}^{m}(1\otimes b_j)\otimes(y_{2m}\otimes b_{m-j})\} \\
= \; & \lambda_* \circ (1\times\iota)_* \{\sum_{j=0}^{m}(b'_{m+j} + b'_m b_j)\otimes b_{m-j} + \rho {b'_m}^2 \otimes 1 \\
& + \delta_m b'_m \otimes b'_m + \sum_{j=0}^{m} b_{m-j}\otimes b'_{m+j}\} \\
= \; & (\rho+\delta_m){b'_m}^2 + \sum_{j=0}^{m} b'_m b_j b_{m-j}.
\end{aligned}$$

(As defined in §2, $\delta_{\text{even}} = 1$ and $\delta_{\text{odd}} = 0$.)

Here remark that, when m is even, $b'_m = {b_{m/2}}^2$ and we obtain

$$\begin{aligned}
\eta'_*(y_{2m}\otimes b_m) &= (\rho+\delta_m){b'_m}^2 + \sum_{j=0}^{m} b'_m b_j b_{m-j} \\
&= (\rho+\delta_m){b'_m}^2 + 2\sum_{0\le j<m/2} b'_m b_j b_{m-j} + \delta_m b'_m {b_{m/2}}^2 \\
&= \rho {b'_m}^2.
\end{aligned}$$

∎

In the following we use integral homology. By the result of R.Bott ([1]), $\mathrm{H}_*(\Omega_0 SO(2m+2);\mathbf{Z})$ has its generators $\overline{b_j}$, $\overline{b'_{m+j}}$ $(0 \le j \le m)$, whose mod 2 reduction is b_j, b'_{m+j} in $\mathrm{H}_*\left(\Omega_0 SO(2m+2); \mathbf{Z}/2\mathbf{Z}\right)$, and its relations are:

$$
\begin{aligned}
0 &= \overline{b_j}^2 + 2\sum_{k=1}^{j}(-1)^k \overline{b_{j-k}b_{j+k}}, && (1 \le j < m/2)\\
0 &= \overline{b_j}^2 + 2\sum_{k=1}^{m-j}(-1)^k \overline{b_{j-k}b_{j+k}} + \sum_{k=m-j}^{j} \overline{b_{j-k}}\overline{b'_{j+k}}. && (m/2 \le j \le m)
\end{aligned}
$$

Also, it can be easily checked that

$$\mathrm{H}^*\left(RP^{2m+1}/RP^{2m-1}; \mathbf{Z}\right) = \mathbf{Z} \oplus \mathbf{Z}\tau_{2m} \oplus \mathbf{Z}\tau_{2m+1}$$

where $|\tau_i| = i$. And if we denote the generator of $\mathrm{H}^1\left(RP^{2m+1}; \mathbf{Z}/2\mathbf{Z}\right)$ by τ, the mod 2 reduction of $p_m^*\tau_i$ is τ^i.

Theorem 5.2. *In Proposition 4.3, $\rho = 0$.*

Proof. First we calculate $\eta_*(\tau_{2m} \otimes \overline{b_m}) \in \mathrm{H}_*\left(\Omega_0 SO(2m+2); \mathbf{Z}\right)$ in integral homology. Since we know $\Delta_*\overline{b_m} = \sum_{i+j=m} \overline{b_i} \otimes \overline{b_j}$ (See [1].), we can compute its coproduct as

$$
\begin{aligned}
& \Delta_* \circ \eta_*(\tau_{2m} \otimes \overline{b_m})\\
= \ & (\eta \times \eta)_* \circ \Delta_*(\tau_{2m} \otimes \overline{b_m})\\
= \ & (\eta \times \eta)_*\Big(\sum_{i+j=m} \tau_{2m} \otimes \overline{b_i} \otimes 1 \otimes \overline{b_j} + \sum_{i+j=m} 1 \otimes \overline{b_i} \otimes \tau_{2m} \otimes \overline{b_j}\Big)\\
= \ & \sum_{i+j=m} \{\eta_*(\tau_{2m} \otimes \overline{b_i}) \otimes \eta_*(1 \otimes \overline{b_j})\} + \sum_{i+j=m} \{\eta_*(1 \otimes \overline{b_i}) \otimes \eta_*(\tau_{2m} \otimes \overline{b_j})\}.
\end{aligned}
$$

Here remark that, since $\eta'|_{SO(2m)\times G^s} \simeq *$, $\eta|_{*\times G^s}$ is null homotopic, too. Thus $\eta_*(1 \otimes \overline{b_j}) = 0$, if $j > 0$ and we obtain that $\eta_*(\tau_{2m} \otimes \overline{b_m})$ is primitive. But there exists no primitive element in $\mathrm{H}_{4m}\left(\Omega_0 SO(2m+2); \mathbf{Z}\right)$. (See Proposition 10.1 in [1].) Therefore $\eta_*(\tau_{2m} \otimes \overline{b_m}) = 0$.

On the other hand, in mod 2 homology, $\eta_*(\tau_{2m} \otimes b_m) = \eta_* \circ (p_m \times 1)_*(\tau^{2m} \otimes b_m) = \eta'_* \circ (j_{2m+2} \times 1)_*(\tau^{2m} \otimes b_m) = \rho {b'_m}^2$.

Since $\mathrm{H}_*\left(\Omega_0 SO(2m+2); \mathbf{Z}\right)$ is free, it follows that $\rho = 0$. ∎

6 $\mathrm{H}_*\left(LG(Spin(N))\right)$ as Hopf algebra

In this section, using Theorem 4.5, we consider the homology ring of $LG\left(Spin(N)\right)$. First, we see the homology ring of $Spin(N)$. Theorem 6.1 and 6.2 are the dual results of Theorem 2.1.

Theorem 6.1. *There are generators y and y_i ($i \in A_N$) of* $\mathrm{H}_*(Spin(N))$ *as algebra where $A_N = \{1, 2, 3, \cdots, N-1\} \backslash \{$powers of 2$\}$, $|y_i| = i$ and $|y| = 2^{s+1} - 1$. (s is the integer such that $2^s < N \leq 2^{s+1}$.) And their fundamental relations are*

$$y_i{}^2 = 0, \quad y^2 = 0, \tag{7}$$

$$[y_i, y_j] = \begin{cases} y & \text{if } i + j = 2^{s+1} - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

For $1 \leq k \leq 2^p$, we set $k_i \in \{0, 1\}$ ($p = 0, 1, \cdots, p$) as

$$k = \sum_{i=0}^{p} k_i 2^i.$$

And for $a, k \in Z$ such that a is odd, $a > 1$, $ak < N - 1$, we set $Y_{(a,k)} \in \mathrm{H}_{ak}(Spin(N))$ as $Y_{(a,k)} = \prod_{i=1}^{p} k_i y_{a2^i}$.

Theorem 6.2. *The coproducts of generators in* $\mathrm{H}_*(Spin(N))$ *are given as*

$$\begin{aligned} \overline{\Delta}_* y_i &= \sum_{\substack{k+k'=2^p \\ k>0, k'>0}} Y_{(a,k)} \otimes Y_{(a,k')}, \\ \overline{\Delta}_* y &= 0 \end{aligned}$$

where $i = a2^p$, a is odd and $a > 1$.

By the diagram in §1, we can calculate $\mathrm{H}_*(LG\,(Spin(N)))$ directly.

By the isomorphism $\mathrm{H}_*(LG\,(Spin(N))) \cong \mathrm{H}_*(\Omega Spin(N)) \otimes \mathrm{H}_*(Spin(N))$ as $\mathbf{Z}/\,2\,\mathbf{Z}$ module we identify $b_i, b'_{m+i} \in \mathrm{H}_*(\Omega Spin(N))$ and $y_i, y \in \mathrm{H}_*(Spin(N))$ with elements of $\mathrm{H}_*(LG\,(Spin(N)))$. Here $|b_i| = 2i$, $|b'_{m+i}| = 2(m+i)$, $|y_i| = i$ and $|y| = 2^{s+1} - 1$ where s is the number difined in Theorem 6.1.

Theorem 6.3. *1. (a) When N is even, we set $N = 2m + 2$. Then $\mathrm{H}_*(LG\,(Spin(N)))$ is the non-commutative Hopf algebra generated by b_i, b'_{m+i} ($i = 0, 1, \cdots, m$), y_j ($j \in A_N$) and y.*

(b) When N is odd, we set $N = 2m+1$. Then $\mathrm{H}_*(LG\,(Spin(N)))$ is the non-commutative Hopf algebra generated by b_i, b'_{m+i} ($i = 0, 1, \cdots, m-1$), y_i ($j \in A_N$) and y.

2. *Their relations are*
   (a) *The relations in* $\mathrm{H}_*\left(Spin(N)\right)$ *shown in Theorem 6.1.*
   (b) *The relations in* $\mathrm{H}_*\left(\Omega Spin(N)\right)$ *shown in Theorem 2.3.*
   (c) *For* $y_j \in \mathrm{H}_*\left(Spin(N)\right)$ *and* $b \in \mathrm{H}_*\left(\Omega Spin(N)\right)$

$$y_j b = \sum (y' * b) y''$$

   where $\Delta_* y_j = \sum y' \otimes y''$ *and each* $y' * b$ *can be obtained from Theorem 4.5.*

3. *The coproducts of these generators in* $\mathrm{H}_*\left(LG\left(Spin(N)\right)\right)$ *are the same as the coproducts of those in* $\mathrm{H}_*\left(\Omega Spin(N)\right)$ *or* $\mathrm{H}_*\left(Spin(N)\right)$ *shown in Theorem 2.4 and Theorem 6.2.*

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