Involution and anti involution on Hopf spaces

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1 Introduction

A pointed space X with a base point preserving map $\mu : X \times X \to X$ is said to be a Hopf space if the restriction of μ to $X \vee X$ is homotopic to $\mathrm{id} \vee \mathrm{id}$. A pointed $\mathbf{Z}/2\mathbf{Z}$ space X is said a Hopf space with involution if X is a Hopf space and the structure map $\mu : X \times X \to X$ is equivariant and the restriction of μ to $X \vee X$ is equivariantly homotopic to $\mathrm{id} \vee \mathrm{id}$.

The concept of a Hopf space came from that of a topological group and a topological group G has an involution τ defined by $\tau(g) = g^{-1}$ for $g \in G$. The product of G is not equivariant but it has the property

$$\tau(xy) = \tau(y)\tau(x).$$

This leads us to the definition of Hopf space with anti involution . That is, a $\mathbf{Z}/2\mathbf{Z}$ space X which is a Hopf space is said to be a Hopf space with anti involution if the structure map $\mu : X \times X \to X$ is equivariant with respect to the involution $\tilde{\tau}$ defined by $\tilde{\tau}(x, y) = (\tau y, \tau x)$, and the restriction of μ to $X \vee X$ is equivariantly homotopic to $\mathrm{id} \vee \mathrm{id}$.

Adams showed that S^n has a structure of Hopf space if and only if $n = 0, 1, 3, 7$. In [4] ,Iriye showed that the unit sphere of a real representation space V of $\mathbf{Z}/2\mathbf{Z}$ admits a structure of a Hopf space with involution if and only if V is $\mathbf{R}^{1,1}$, $\mathbf{R}^{2,2}$, $\mathbf{R}^{4,4}$, $\mathbf{R}^{0,1}$, $\mathbf{R}^{0,2}$, $\mathbf{R}^{0,4}$ or $\mathbf{R}^{0,8}$.

In this paper we shall show that a $\mathbf{Z}/2\mathbf{Z}$ homology sphere with involution of which 'type' has a structure of Hopf space with involution or anti involution. From localization theorem we have that the fixed point set of a $\mathbf{Z}/2\mathbf{Z}$ homology sphere with involution is also a $\mathbf{Z}/2\mathbf{Z}$ homology sphere. Therefore we say that a $\mathbf{Z}/2\mathbf{Z}$ homology n sphere whose fixed point set is a $\mathbf{Z}/2\mathbf{Z}$ m sphere is of type (n, m).

Theorem Let $d = 1$, 2 or 4. There exists a $\mathbf{Z}/2\mathbf{Z}$ space X of type $(2d-1, p)$ which is a Hopf space with involution, if and only if $p = d-1, 2d-1$. There exists a $\mathbf{Z}/2\mathbf{Z}$ space X of type $(2d-1, p)$ which is a Hopf space with anti involution, if and only if $p = 0, d$.

See theorem 5.12 and 5.13.

This paper is organized as follows: In §2 we define Hopf space with involution or anti involution and offer examples. In §3 we introduce the Hopf construction of a Hopf space with involution or anti iovolution and show that its Hopf invariant is one. Also localization theorem plays an important role in this paper. Thus we refer to this theorem in §4. Then in §5, we shall show when an equivariant map from S^{4d-1} to S^{2d} with involutions has Hopf invariant one. This leads us to previous theorem.

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2 Hopf space with involution, anti involution

Let $(X\ \mu)$ be a Hopf space. Suppose that X has an involution τ,that is, X is a $\mathbf{Z}/2\mathbf{Z}$ space.

If $\tau\mu(x, y) = \mu(\tau x, \tau y)$ for all $x, y \in X$ and the restriction of μ to $X \vee X$ is equivariantly homotopic to $id \vee id$,then we say (X, μ, τ) is a Hopf space with involution .

If $\tau\mu(x, y) = \mu(\tau y, \tau x)$ for all $x, y \in X$ and the restriction of μ to $X \vee X$ is equivariantly homotopic to $id \vee id$ where the involution of $X \times X$ is defined by

$$\tilde{\tau}(x, y) = (\tau y, \tau x),$$

then we say (X, μ, τ) is a Hopf space with anti involution .

Example 2.1 *The unit spheres of* $\mathbf{R}^{1,1}$,$\mathbf{R}^{2,2}$,$\mathbf{R}^{4,4}$ *are Hopf spaces with involution. See Iriye [4].*

Example 2.2 *Let G be a Lie group. Then the ordinary product of G makes G a Hopf space. G has an involution $\tau : G \to G$ defined by $\tau(x) = x^{-1}$. Thus G with τ is a Hopf space with anti involution .*

Example 2.3 *$GL(n, K)$ with the involution $\tau : GL(n, K) \to GL(n, K)$ defined by $\tau(A) = {}^tA$ is a Hopf space with anti involution by the ordinary product.*

Example 2.4 *We regard S^3(resp. S^7) as the unit sphere in* $\mathbf{H}$ *(resp. the Cayley numbers* $\mathbf{O}$*). Then the involution τ defined by $\tau(x) = \bar{x}$ makes S^3(resp. S^7) a Hopf space with anti involution by the ordinary product in* $\mathbf{H}$ *(resp.* $\mathbf{O}$*).*

3 Hopf construction with involution

Given a map $\mu : A \times B \to C$,the Hopf construction $H(\mu) : A * B \to \Sigma C$ is defined by $H(\mu)(a, t, b) = (t, \mu(a, b))$,where $a \in A, b \in B, t \in [0, 1]$.

Suppose that (X,μ,τ) is a Hopf space with involution or anti involution. We introduce involutions to $X * X, \Sigma X$ so as to make $H(\mu) : X * X \to \Sigma X$ equivariant. We define involutions $\tau_0{}', \tau_1{}' : X * X \to X * X$ and $\tau_0'', \tau_1'' : \Sigma X \to \Sigma X$ as follows:

$$\tau_0'(x, t, y) = (\tau x, t, \tau y) \quad \tau_1'(x, t, y) = (\tau y, 1 - t, \tau x) \quad x, y \in X, t \in [0, 1]$$

$$\tau_0''(t, x) = (t, \tau x) \quad \tau_1''(t, x) = (1 - t, \tau x) \quad x \in X, t \in [0, 1]$$

If (X,μ,τ) is a Hopf space with involution then $H(\mu) : (X*X, \tau_0') \to (\Sigma X, \tau_0'')$ is equivariant. If (X,μ,τ) is a Hopf space with anti involution then $H(\mu) : (X * X, \tau_1') \to (\Sigma X, \tau_1'')$ is equivariant.

Now (X,μ) is a Hopf space and let $f = H(\mu)$. In the following we use $\mathbf{Z}/2\mathbf{Z}$ as the coefficient ring of cohomology rings unless mentioned.

Theorem 3.5 *Stasheff [5] Let (X,μ) be a Hopf space and C_f be the mapping cone of $f = H(\mu)$. Consider the next exact sequence.*

$$\to \mathrm{H}^*(\Sigma(X * X)) \to \mathrm{H}^*(C_f) \to \mathrm{H}^*(\Sigma X) \to \mathrm{H}^*(X * X)$$

If $u, v \in \mathrm{H}^*(X)$ are classes such that $\Sigma u, \Sigma v \in \mathrm{H}^*(\Sigma X)$ pull back to $\mathrm{H}^*(C_f)$ then $\Sigma u \cup \Sigma v \in \mathrm{H}^*(C_f)$ comes from $\Sigma(u * v) \in \mathrm{H}^*(\Sigma(X * X))$ which is isomorphic to $\mathrm{H}^*(C_f, \Sigma X)$.

Theorem 3.6 *Let $n > 1$. If σ^{n-1} is a $\mathbf{Z}/2\mathbf{Z}$ homology $n-1$ sphere and (σ^{n-1}, μ) is a Hopf space, then $\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$ where degree of x is n.*

For $n = 1$,$\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$ or $\mathbf{Z}/2\mathbf{Z}$.

Proof Consider the next exact sequence.

$$\to \widetilde{\mathrm{H}}^*(\Sigma(\sigma^{n-1} * \sigma^{n-1})) \to \widetilde{\mathrm{H}}^*(C_f) \to \widetilde{\mathrm{H}}^*(\Sigma\sigma^{n-1}) \to \widetilde{\mathrm{H}}^*(\sigma^{n-1} * \sigma^{n-1})$$

And remark $\mathrm{H}^*(\sigma^{n-1} * \sigma^{n-1}) \cong \mathrm{H}^*(S^{2n-1})$, $\mathrm{H}^*(\Sigma\sigma^{n-1}) \cong \mathrm{H}^*(S^n)$, $n \neq 2n-1$. Then we have

$$\mathrm{H}^*(C_f) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & * = 0, n, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Let u be the generator of $\mathrm{H}^{n-1}(\sigma^{n-1})$ then Σu comes from x the generator of $\mathrm{H}^n(C_f)$ and $x \cup x$ comes from $\Sigma(u * u)$ the generator of $\mathrm{H}^{2n}(\Sigma(\sigma^n * \sigma^n))$. Hence $x \cup x$ is the generator of $\mathrm{H}^{2n}(C_f)$.

For the latter part of the theorem, consider the same exact sequence and remark that $n = 2n - 1 = 1$.

$$\to \widetilde{\mathrm{H}}^*(\Sigma(\sigma^{n-1} * \sigma^{n-1})) \to \widetilde{\mathrm{H}}^*(C_f) \to \widetilde{\mathrm{H}}^*(\Sigma\sigma^{n-1}) \to \widetilde{\mathrm{H}}^*(\sigma^{n-1} * \sigma^{n-1})$$

We obtain that f^* is a 0 map or an isomorphism. Hence, $\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$ or $\mathbf{Z}/2\mathbf{Z}$ respectively.

Remark Let $n > 1$. Given $f : \sigma^{2n-1} \to \sigma^n$ where σ^i's are $\mathbf{Z}/2\mathbf{Z}$ homology i spheres ($i = n, 2n-1$),then

$$\mathrm{H}^*(C_f) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & * = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

and let x, y be the generator of $H^n(C_f), \mathrm{H}^{2n}(C_f)$ respectively. We define the Hopf invariant $\gamma(f) \in \mathbf{Z}/2\mathbf{Z}$ by $x^2 = \gamma(f)y$.

4 Localization theorem

For a compact Lie group G and a G space X with a fixed base point $*$, we define $\mathrm{H}^*_G(X;\Lambda)$ as follows. Let $\mathbf{PG} \to \mathbf{BG}$ be the universal G bundle.

$$\mathrm{H}^*_G(X;\Lambda) \equiv \mathrm{H}^*(\mathbf{PG}\times_G X;\Lambda)$$

where Λ is a ring. The reduced equivariant colomology $\widetilde{\mathrm{H}}^*_G(X;\Lambda)$ is defined as follows:

$$\begin{aligned}\widetilde{\mathrm{H}}^*_G(X;\Lambda) &\equiv \mathrm{H}^*(\mathbf{PG}\times_G X, \mathbf{PG}\times_G *;\Lambda)\\ &= \ker s^*\end{aligned}$$

where s is the section of $\mathbf{PG}\times_G X \to \mathbf{BG}$ defined by

$$s(x) = (y,*) \text{ where } x \in \mathbf{BG}, y \in p^{-1}(x).$$

In the following we only consider the case $G = \mathbf{Z}/2\mathbf{Z}$ and $\Lambda = \mathbf{Z}/2\mathbf{Z}$. In that case, $\mathrm{H}^*_G \equiv \mathrm{H}^*(\mathbf{BG}) = \mathbf{Z}/2\mathbf{Z}[t]$.

Remark If G acts on X trivially, then $\mathbf{PG}\times_G X = \mathbf{BG} \times X$. Hence $\mathrm{H}^*_G(X) = \mathrm{H}^*(X)\otimes_{\mathbf{Z}/2\mathbf{Z}}\mathrm{H}^*_G$.
We refer to the next theorem (Quillen [3]).

Theorem 4.7 *(localization theorem) If X is a compact G space, then the inclusion i of the fixed point set X^G into X induces an isomorphism*

$$\widetilde{\mathrm{H}}^*_G(X)[t^{-1}] \xrightarrow{\sim} \widetilde{\mathrm{H}}^*_G(X^G)[t^{-1}]$$

where $\widetilde{\mathrm{H}}^*_G(X)[t^{-1}]$ means the localization of $\widetilde{\mathrm{H}}^*_G(X)$ by t^{-1}.

From localization theorem we obtain two propositions.

Proposition 4.8 *If X is a compact G space and $\mathrm{H}^*(X;\mathbf{Z}/2\mathbf{Z}) \cong \mathrm{H}^*(S^l;\mathbf{Z}/2\mathbf{Z})$ for some $l \geq 0$ and $X^G \neq \phi$, then $\mathrm{H}^*(X^G) \cong \mathrm{H}^*(S^m)$ for some $m \leq l$.*

Proof See Bredon [2].

Proposition 4.9 *Let X be a compact G space. If $X^G \neq \phi$ and $\mathrm{H}^*(X;\mathbf{Z}/2\mathbf{Z})$ is generated by one element as a graded $\mathbf{Z}/2\mathbf{Z}$ algebra,then*

$$i^* : \widetilde{\mathrm{H}}^*_G(X) \to \widetilde{\mathrm{H}}^*_G(X^G) \text{ is monic.}$$

Proof First we prove that $\widetilde{\mathrm{H}}^*_G(X)$ is a free H^*_Gmodule. Consider the Serre spectral sequence $E_i^{j,k}$ of the fiber space $\mathbf{PG}\times_G X \to \mathbf{BG}$.

Let x denote the generator of $\mathrm{H}^*(X)$ and the degree of x be m. Then

$$E_2^{p,q} = 0 \ \ (q \notin m\mathbf{Z}).$$

Also, X has a fixed point and $p : \mathbf{PG}\times_G X \to \mathbf{BG}$ has a section s. Thus $p^* : \mathrm{H}^*(\mathbf{BG}) \to \mathrm{H}^*(\mathbf{PG}\times_G X)$ is monic. The image of p^* is $\bigoplus_{p\geq 0} E_\infty^{p,0}$. Therefore we have that

$$d_m^{0,m}(1 \otimes x) = 0.$$

Since x generates $\mathrm{H}^*(X)$, the Serre spectral sequence is trivial. Therefore $\mathrm{H}^*_G(X)$ is a free H^*_Gmodule. $\widetilde{\mathrm{H}}^*_G(X)$ is a submodule of $\mathrm{H}^*_G(X)$ and it follows that $\widetilde{\mathrm{H}}^*_G(X)$ is a free H^*_Gmodule.

Then consider the following commutative diagram .

$$\begin{array}{ccc} \widetilde{\mathrm{H}}^*_G(X) & \to & \widetilde{\mathrm{H}}^*_G(X)[t^{-1}] \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{H}}^*_G(X^G) & \to & \widetilde{\mathrm{H}}^*_G(X)[t^{-1}] \end{array}$$

In the diagram the arrows which goes down means i^* and the arrows which goes adross are monic since $\widetilde{\mathrm{H}}^*_G(X)and\widetilde{\mathrm{H}}^*_G(X^G)$ are free H^*_Gmodules. Hence by localization theorem we have that $i^* : \widetilde{\mathrm{H}}^*_G(X) \to \widetilde{\mathrm{H}}^*_G(X^G)$ is monic.

5 Proof of main theorem

Let $\mathcal{S}^{n,m}$ mean the set of G isomorphism classes of all $\mathbf{Z}/2\mathbf{Z}$ homology n spheres which are compact G spaces and whose fixed point sets are $\mathbf{Z}/2\mathbf{Z}$ homology m spheres. (Propsition 4.8 says that if X is a compact $\mathbf{Z}/2\mathbf{Z}$ space and at the same time a $\mathbf{Z}/2\mathbf{Z}$ homology n sphere, then $X \in \mathcal{S}^{n,m}$ for some m.)

Theorem 5.10 *Let $d = 1$, 2 or 4. There exist $\sigma^{4d-1,q} \in \mathcal{S}^{4d-1,q}, \sigma^{2d,q'} \in \mathcal{S}^{2d,q'}$ and a continuous Gmap $f : \sigma^{4d-1,q} \to \sigma^{2d,q'}$ such that*

$$\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \ \textit{where}\ |x| = 2d,$$

that is,the Hopf invariant is one, if and only if $(q, q') = (4d-1, 2d)$ or $(2d-1, d)$ or $(2d-1, 0)$.

Remark Adams showed that

> If $\mathrm{H}^*(X) \cong \mathrm{H}^*(S^{2n-1})$ and $\mathrm{H}^*(Y) \cong \mathrm{H}^*(S^n), n \geq 1$,and there exists $f : X \to Y$ such that $\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$, then $n = 1, 2, 4, 8$.

Therefore the previous theorem tells about the equivariant case of $n = 2, 4, 8$. For the equivariant case of $n = 1$ see theorem 5.11.

Proof First we assume that there is a continuous G map $f : \sigma^{4d-1,q} \to \sigma^{2d,q'}$ such that $\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3), |x| = 2d$,where $\sigma^{4d-1,q} \in \mathcal{S}^{4d-1,q}$ and $\sigma^{2d,q'} \in \mathcal{S}^{2d,q'}$.

By the proof of proposition 4.9, $\widetilde{\mathrm{H}}^*_G(C_f)$ and $\widetilde{\mathrm{H}}^*_G({C_f}^G)$ are free H^*_Gmodules and from the localization theorem the ranks of $\widetilde{\mathrm{H}}^*_G(C_f)$ and $\widetilde{\mathrm{H}}^*_G({C_f}^G)$ are same. Note that $\widetilde{\mathrm{H}}^*_G({C_f}^G) \cong \widetilde{\mathrm{H}}^*({C_f}^G) \otimes \mathrm{H}^*_G$. Therefore we have that

$$\widetilde{\mathrm{H}}^*({C_f}^G) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}. \tag{1}$$

Denotes the restricted map of f to G fixed point set by f^G. Then we have an exact sequence.

$$\to \widetilde{\mathrm{H}}^*(\Sigma^{0,1}{\sigma^{2d,q'}}^G) \xrightarrow{\Sigma f^{G*}} \widetilde{\mathrm{H}}^*(\Sigma^{0,1}{\sigma^{4d-1,q}}^G) \xrightarrow{k^*} \widetilde{\mathrm{H}}^*(C_{f^G}) \xrightarrow{j^*} \widetilde{\mathrm{H}}^*({\sigma^{2d,q'}}^G) \xrightarrow{f^{G*}} \widetilde{\mathrm{H}}^*({\sigma^{4d-1,q}}^G)$$

Here $\Sigma^{0,1}{\sigma^{2d,q'}}^G, \Sigma^{0,1}{\sigma^{4d-1,q}}^G, {\sigma^{2d,q'}}^G$ and ${\sigma^{4d-1,q}}^G$ are $\mathbf{Z}/2\mathbf{Z}$ homology $q' + 1, q + 1, q', q$ spheres respectively and $C_{f^G} = {C_f}^G$. Then from (1) f^{G*} and Σf^{G*} are 0 maps. Now we obtain that

$$\widetilde{\mathrm{H}}^*({C_f}^G) \cong \widetilde{\mathrm{H}}^{q'}(\Sigma^{0,1}{\sigma^{4d-1,q}}^G) \oplus \widetilde{\mathrm{H}}^{q+1}({\sigma^{2d,q'}}^G).$$

Let y be the image of the generator of $\widetilde{\mathrm{H}}^{q'}(\Sigma^{0,1}{\sigma^{4d-1,q}}^G)$ and y' be a pull back of the generator of $\widetilde{\mathrm{H}}^{q+1}({\sigma^{2d,q'}}^G)$. Therefore $\widetilde{\mathrm{H}}^*({C_f}^G)$ is a $\mathbf{Z}/2\mathbf{Z}$ vector space generated by y and y'. We consider y and y' to be elements of $\widetilde{\mathrm{H}}^*_G({C_f}^G)$ by

$$\widetilde{\mathrm{H}}^*_G({C_f}^G) \cong \widetilde{\mathrm{H}}^*({C_f}^G) \otimes \mathrm{H}^*_G \text{ as an algebra.}$$

And also we can consider x, x^2 to be elements of $\widetilde{\mathrm{H}}^*_G(C_f)$ by the isomorphisms

$$\widetilde{\mathrm{H}}^*_G(C_f) \cong \bigoplus_{q>0,p\geq 0} E_\infty^{p,q} \cong \bigoplus_{q>0,p\geq 0} E_2^{p,q} \text{ as an } \mathrm{H}^*_G\text{module}$$

$$\bigoplus_{q>0} E_2^{0,q} \cong \widetilde{\mathrm{H}}^*(C_f) \text{ as a } \mathbf{Z}/2\mathbf{Z} \text{ module.}$$

Here x, x^2 are the basis of $\widetilde{\mathrm{H}}^*_G(C_f)$ as an H^*_Gmodule.

$$\widetilde{\mathrm{H}}^*_G(C_f) \ni \begin{cases} x & * = 2d \\ x^2 & * = 4d \end{cases}$$

$$\widetilde{\mathrm{H}}^*_G({C_f}^G) \ni \begin{cases} y & * = q' \\ y' & * = q+1 \end{cases}$$

First,it is easily seen that $y'^2 = 0$ since

$$\begin{aligned} y'^2 &= k^*((k^*)^{-1}y)^2 \\ &= k^*0 = 0. \end{aligned}$$

Now soppose $i^*(x) = at^{2d-q'}y + bt^{2d-(q+1)}y'$ where $a, b \in \mathbf{Z}/2\mathbf{Z}, \mathrm{H}^*_G = \mathbf{Z}/2\mathbf{Z}[t], i : {C_f}^G \to C_f$. Then we have that

$$\begin{aligned} i^*(x^2) &= i^*(x)^2 \\ &= at^{4d-2q'}y^2 + bt^{4d-2(q+1)}y'^2 \\ &= at^{4d-2q'}y^2. \end{aligned}$$

While proposition 4.9 says i^* is monic. Hence $a \neq 0$ and $y^2 \neq 0$. Thus we have

$$a = 1 \text{ and } 2\deg y = q' \text{ or } q+1.$$

First case We consider the first case $q' = 0$.

If we assume $b = 0$,

$$\begin{aligned} i^*(x) &= t^{2d}y \\ i^*(x^2) &= t^{4d}y^2 = t^{4d}y. \end{aligned}$$

Thus Im i^* is contained in the H_G^*submodule generated by y while localization theorem says $i^* : \widetilde{\mathrm{H}}_G^*(C_f)[t^{-1}] \to \widetilde{\mathrm{H}}_G^*({C_f}^G)[t^{-1}]$ is an isomorphism. Therefore $b = 1$ and we have that

$$i^*(x) = t^{2d}y + t^{2d-(q+1)}y' \text{ and } 2d \geq q+1.$$

Here assume that $q+1 \neq 2d$. Then it follows that

$$\begin{aligned} \mathrm{Sq}^{2d-(q+1)}(i^*x) &= \sum_{i+j=2d-(q+1)} \mathrm{Sq}^i(t^{2d})\mathrm{Sq}^j(y) + t^{4d-2(q+1)}y' \\ &= t^{4d-2(q+1)}y' \text{ since } 2d = 2^p. \end{aligned}$$

$$\begin{aligned} \text{While } i^*(\mathrm{Sq}^{2d-(q+1)}x) &= i^*(0 \text{ or } t^{2d-(q+1)}x) \\ &= 0 \text{ or } t^{4d-(q+1)}y + t^{4d-2(q+1)}y'. \end{aligned}$$

Previous two equations contradict each other. Hence we have $q+1 = 2d$,that is,$q' = 0, q = 2d-1$.

Second case Next we assume $2q' = q+1$. Then $(q+1) - q' = q' \leq 2d, y^2 = y'$ and $i^*(x) = t^{2d-q'}y + bt^{2d-2q'}y^2$.

a) Now assume that $b = 0$, i.e., $i^*(x) = t^{2d-q'}y$. Then

$$\begin{aligned} i^*(\mathrm{Sq}^{q'}(x)) &= \mathrm{Sq}^{q'}(i^*x) \\ &= \mathrm{Sq}^{q'}(t^{2d-q'}y) \\ &= \binom{2d-q'}{q'}t^{2d}y + t^{2d-q'}y^2 \end{aligned} \tag{2}$$

Here if we suppose that $q' \neq 2d$ $(0 \leq q' \leq 2d)$, then $\mathrm{Sq}^{q'}(x) = 0$ or $t^{q'}x$. Thus

$$i^*(\mathrm{Sq}^{q'}x) = 0 \text{ or } t^{2d}y$$

and this contradicts to (2). Therefore we obtain

$$q' = 2d, q = 4d-1.$$

b) The last case is that $i^*(x) = t^{2d-q'}y + t^{2d-2q'}y^2$. Here $2d - 2q' \geq 0$, that is, $d \geq q'$.

Apply the Adams' theorem in remark in §5 to the sequence

$$\sigma^{4d-1,q^G} \to \sigma^{2d,q'^G} \to C_{f^G} \to \Sigma^{0,1}\sigma^{4d-1,q^G} \to \Sigma^{0,1}\sigma^{2d,q'^G} \to$$

$$\text{where } q = 2q' - 1, \mathrm{H}^*(C_{f^G}) = \mathbf{Z}/2\mathbf{Z}[y]/(y^3).$$

And we obtain that $q = 2^r$ for some $r \geq 0$.

Compute $i^*(\mathrm{Sq}^{2q'}(x))$ and we have that

$$\begin{aligned} i^*(\mathrm{Sq}^{2q'}(x)) &= \mathrm{Sq}^{2q'}(t^{2d-q'}y + t^{2d-2q'}y^2) \\ &= \binom{2d-q'}{2q'}t^{2d+q'}y + \binom{2d-q'}{q'}t^{2d}y^2 + \binom{2d-2q'}{2q'}t^{2d}y^2 \\ &= \binom{2d-2q'}{2q'}t^{2d+q'}y + \{\binom{2d-q'}{q'} + \binom{2d-2q'}{2q'}\}t^{2d}y^2. \end{aligned}$$

If we suppose that $q' \neq d (0 < q' \leq d)$, then $\mathrm{Sq}^{2q'}(x) = 0$ or $t^{2q'}x$. And it follows that

$$i^*(\mathrm{Sq}^{2q'}(x)) = 0 \text{ or } t^{2d+q'}y + t^{2d}y^2.$$

Hence we obtain

$$\binom{2d-q'}{2q'} = \binom{2d-q'}{q'} + \binom{2d-2q'}{2q'} \text{ mod } 2.$$

Let $2d = 2^p (p \geq r+1)$ and we have

$$\begin{aligned} \binom{2^p - 2^r}{2^{r+1}} &= \binom{2^p - 2^r}{2^r} + \binom{2^p - 2^{r+1}}{2^{r+1}} \text{ mod } 2 \\ \binom{2^{p-r} - 1}{2} &= \binom{2^{p-r}}{1} + \binom{2^{p-r-1}}{1} \text{ mod } 2 \\ (2^{p-r} - 1)(2^{p-r-1} - 1) &= 0 \text{ mod } 2 \end{aligned}$$

Therefore we obtain

$$p - r = 0 \text{ or } p - r - 1 = 0.$$

But $p \geq r+1$. Hence $p = r+1$, i.e.,

$$q' = d, q = 2d - 1.$$

We complete the proof of the former part of theorem.

All we have to do is to show the existence of $\sigma^{4d-1,q} \in \mathcal{S}^{4d-1,q}, \sigma^{2d,q'} \in \mathcal{S}^{2d,q'}$ and a continuous G map $f : \sigma^{4d-1,q} \longrightarrow \sigma^{2d,q'}$ such that

$$\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \text{ where } |x| = 2d,$$

for $(q,q')=(4d-1,2d)$, $(2d-1,d)$, $(2d-1,0)$. We construct these in theorem 5.12,5.13. Q.E.D.

Remark A part of previous theorem can be proved by using Bredon's theorem [2, pp.425–427 theorem 11.1]. But his proof uses a not obvious fact. Thus we offered our own proof.

Theorem 5.11 *There exist $\sigma^{1,q}\in\mathcal{S}^{1,q},\sigma^{1,q'}\in\mathcal{S}^{1,q'}$ and a continuous Gmap $f:\sigma^{1,q}\longrightarrow\sigma^{1,q'}$ such that*

$$\mathrm{H}^*(C_f)=\begin{array}{l}\mathbf{Z}/2\mathbf{Z}[x]/(x^3)\ \textit{where}\ |x|=1\\ \textit{or}\ \mathbf{Z}/2\mathbf{Z}\end{array}$$

,if and only if $(q,q')=(1,1)$ or $(0,0)$.

Proof First we assume that there are $\sigma^{1,q}\in\mathcal{S}^{1,q},\sigma^{1,q'}\in\mathcal{S}^{1,q'}$ and a continuous G map $f:\sigma^{1,q}\longrightarrow\sigma^{1,q'}$ such that

$$\mathrm{H}^*(C_f)=\mathbf{Z}/2\mathbf{Z}[x]/(x^3)\text{ where }|x|=1.$$

a)We suppose $(q,q')=(0,1)$. Consider the exact sequence

$$\to\widetilde{\mathrm{H}}^*(\Sigma^{0,1}\sigma^{1,1^G})\overset{\Sigma f^{G*}}{\to}\widetilde{\mathrm{H}}^*(\Sigma^{0,1}\sigma^{1,0^G})\overset{k^*}{\to}\widetilde{\mathrm{H}}^*(C_{f^G})\overset{j^*}{\to}\widetilde{\mathrm{H}}^*(\sigma^{1,1^G})\overset{f^{G*}}{\to}\widetilde{\mathrm{H}}^*(\sigma^{1,0^G}).$$

It is easily seen that f^* and Σf^{G*} are 0 maps and that

$$\widetilde{\mathrm{H}}^*(C_f{}^G)\cong\begin{cases}\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z} & *=1\\ 0 & \text{otherwise}\end{cases}.$$

Let y,y' be the basis of $\widetilde{\mathrm{H}}^1(C_f{}^G)$. Consider y and y' to be elements of $\widetilde{\mathrm{H}}^*_G(C_f{}^G)$ and x,x^2 to be elements of $\widetilde{\mathrm{H}}^*_G(C_f)$ as we did in the proof of theorem 5.10.

Let $i^*(x)=ay+by'$ where $a,b\in\mathbf{Z}/2\mathbf{Z}$. Then $i^*(x^2)=0$. This contradicts to proposition 4.9. Thus $(q,q')\neq(0,1)$.

b) Next we suppose $(q,q')=(1,0)$. Consider the exact sequence

$$\to\widetilde{\mathrm{H}}^*(\Sigma^{0,1}\sigma^{1,0^G})\overset{\Sigma f^{G*}}{\to}\widetilde{\mathrm{H}}^*(\Sigma^{0,1}\sigma^{1,1^G})\overset{k^*}{\to}\widetilde{\mathrm{H}}^*(C_{f^G})\overset{j^*}{\to}\widetilde{\mathrm{H}}^*(\sigma^{1,0^G})\overset{f^{G*}}{\to}\widetilde{\mathrm{H}}^*(\sigma^{1,1^G}).$$

It is easily seen that f^* and Σf^{G*} are 0 maps and that

$$\widetilde{\mathrm{H}}^*(C_f{}^G)\cong\begin{cases}\mathbf{Z}/2\mathbf{Z} & *=0,2\\ 0 & \text{otherwise.}\end{cases}$$

Let y, y' be the generator of $\widetilde{\mathrm{H}}^0({C_f}^G)$ and $\widetilde{\mathrm{H}}^2({C_f}^G)$ respectively and consider y and y' to be elements of $\widetilde{\mathrm{H}}^*_G({C_f}^G)$, x and x^2 to be elements of $\widetilde{\mathrm{H}}^*_G(C_f)$.

Let $i^*(x) = aty$ where $a \in \mathbf{Z}/2\mathbf{Z}$. Then $i^*(x^2) = a^2t^2y$. This contradicts to localization theorem. Thus $(q, q') \neq (1, 0)$.

Next we assume that there is $\sigma^{1,q} \in \mathcal{S}^{1,q}, \sigma^{1,q'} \in \mathcal{S}^{1,q'}$ and a continuous G map $f : \sigma^{1,q} \to \sigma^{1,q'}$ such that

$$\widetilde{\mathrm{H}}^*(C_f) = 0.$$

a') Just as we have seen in a),

$$\widetilde{\mathrm{H}}^*({C_f}^G) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\widetilde{\mathrm{H}}^*_G({C_f}^G) \neq 0$. This contradicts to localization theorem.

b') Just as b),

$$\widetilde{\mathrm{H}}^*({C_f}^G) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\widetilde{\mathrm{H}}^*_G({C_f}^G) \neq 0$. This contradicts to localization theorem.

Actually we construct $f : \sigma^{1,q} \to \sigma^{1,q'}$ $(q, q') = (1, 1), (0, 0)$ as follows. Let $S^1 = \{z \in \mathbf{C} || z| = 1\}$ and τ denote the involution of S^1 by conjugation. Here $(S^1, \mathrm{id}_{S^1}) \in \mathcal{S}^{1,1}$ and $(S^1, \tau) \in \mathcal{S}^{1,0}$. Let f_0, f_1 be maps from S^1 to S^1 defined by

$$f_0(z) = z,\ f_1(z) = z^2.$$

Both of f_0, f_1 are equivariant whether S^1 takes the trivial involution or the involution τ.

It follows easily

$$C_{f_0} \cong D^2, C_{f_1} \cong \mathbf{RP}^2.$$

Thus $\mathrm{H}^*(C_{f_0}) \cong \mathbf{Z}/2\mathbf{Z}, \mathrm{H}^*(C_{f_1}) \cong \mathbf{Z}/2\mathbf{Z}[x]/(x^3)$ where $|x| = 1$.

Theorem 5.12 *Let $d = 1$, 2 or 4. There exists $\sigma^{2d-1,p} \in \mathcal{S}^{2d-1,p}$ which has a structure of Hopf space with involution,if and only if $p = d-1, 2d-1$.*

Actually the unit sphere of $\mathbf{R}^{0,2d}, \mathbf{R}^{d,d}$ have structures of Hopf space with involution.

Proof First we show that the unit spheres of $\mathbf{R}^{0,2d}$ and $\mathbf{R}^{d,d}$ are Hopf space with involution.

Let $S^{n,m}$ denote the unit sphere of $\mathbf{R}^{n,m}$. It is trivial that $S^{0,2d}$ is a Hopf space with involution . Hence we consider $S^{d,d}$.

$\mathbf{R}^{d,d}$ is identified with $\mathbf{L} \cong \mathbf{K} \oplus \mathbf{K}\omega$ where $\omega^2 = -1$, $\mathbf{L} = \mathbf{C}, \mathbf{H}$, Cayley numbers $\mathbf{O}$, $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ for $d = 1, 2, 4$ respectively. And also $\tau|_{\mathbf{K}} = id, \tau|_{\mathbf{K}\omega} = -1$. With this involution the natural product μ of $\mathbf{C}, \mathbf{H}$, Cayley numbers $\mathbf{O}$ becomes a equivariant map. And $S^{d,d}$ with this product is a Hopf space with involution . See Iriye [4].

Consider the Hopf constructions with involution of $S^{0,2d}$,$S^{d,d}$,and we obtain the existence of $\sigma^{4d-1,q} \in \mathcal{S}^{4d-1,q}, \sigma^{2d,q'} \in \mathcal{S}^{2d,q'}$ and a continuous Gmap $f : \sigma^{4d-1,q} \to \sigma^{2d,q'}$ such that

$$\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \text{ where } |x| = 2d,$$

for $(q, q') = (4d - 1, 2d)$ and $(2d - 1, d)$.

For the former part of the proposition consider the Hopf construction with involution and apply theorem 5.10.

Assume that $\sigma^{2d-1,p} \in \mathcal{S}^{2d-1,p}$ is a Hopf space with involution. Let f be the Hopf construction of the Hopf structure μ.

$$f : \sigma^{2d-1,p} * \sigma^{2d-1,p} \to \Sigma^{0,1}\sigma^{2d-1,p}$$

Remark that

$$\begin{aligned} \sigma^{2d-1,p} * \sigma^{2d-1,p} &\in \mathcal{S}^{4d-1,2p+1} \\ \Sigma^{0,1}\sigma^{2d-1,p} &\in \mathcal{S}^{2d,p+1}. \end{aligned}$$

Hence theorem 5.10 says

$$(2p + 1, p + 1) = (4d - 1, 2d) \text{ or } (2d - 1, d) \text{ or } (2d - 1, 0).$$

The solutions without contradictions are $p = d - 1, 2d - 1$.

Theorem 5.13 *Let $d = 1$, 2 or 4. There exists $\sigma^{2d-1,p} \in \mathcal{S}^{2d-1,p}$ which is a Hopf space with anti involution,if and only if $p = 0, d$.*

Actually the unit spheres of $\mathbf{R}^{2d-1,1}, \mathbf{R}^{d-1,d+1}$ are Hopf spaces with anti involution.

Proof First we show that $S^{2d-1,1}$,$S^{d-1,d+1}$ have structures of Hopf space with anti involution.

Identify $\mathbf{R}^{2d}$ with $\mathbf{L} \cong \mathbf{K} \oplus \mathbf{K}\omega$ where $\omega^2 = -1$, $\mathbf{L} = \mathbf{C}, \mathbf{H}$, Cayley numbers $\mathbf{O}$, $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ for $d = 1, 2, 4$ respectively. We introduce liner involutions τ_0, τ_1 of $\mathbf{K} \oplus \mathbf{K}\omega$ as follows

$$\begin{aligned} \tau_0(x + y\omega) &= \bar{x} + y\omega \\ \tau_1(x + y\omega) &= \bar{x} - y\omega \text{ for } x, y \in \mathbf{K}. \end{aligned}$$

Here the standard product of $\mathbf{C}, \mathbf{H}$, Cayley numbers $\mathbf{O}$ has the property

$$\tau_i(zw) = \tau_i(w)\tau_i(z) \text{ for } z, w \in \mathbf{L}, i = 0, 1.$$

Identify $S^{d-1,d+1}$,$S^{2d-1,1}$ with the unit spheres of $(\mathbf{R}^{2d}, \tau_0), (\mathbf{R}^{2d}, \tau_1)$ respectively and then $S^{d-1,d+1}$,$S^{2d-1,1}$ become Hopf space with anti involution with the standard product of $\mathbf{L}$.

Consider the Hopf constructions with anti involution of $S^{2d-1,1}$,$S^{d-1,d+1}$ and we obtain the existence of $\sigma^{4d-1,q} \in \mathcal{S}^{4d-1,q}$, $\sigma^{2d,q'} \in \mathcal{S}^{2d,q'}$ and a continuous G map $f : \sigma^{4d-1,q} \to \sigma^{2d,q'}$ such that

$$\mathrm{H}^*(C_f) = \mathbf{Z}/2\mathbf{Z}[x]/(x^3) \text{ where } |x| = 2d,$$

for $(q, q') = (2d-1, 0)$ and $(2d-1, d)$.

For the former part of the proposition consider the Hopf construction with anti involution and apply theorem 5.10.

Assume that $\sigma^{2d-1,p} \in \mathcal{S}^{2d-1,p}$ has a structure of Hopf space with anti involution. Let f be the Hopf construction of the Hopf structure μ.

$$f : \sigma^{2d-1,p} * \sigma^{2d-1,p} \to \Sigma^{0,1}\sigma^{2d-1,p}$$

Remark that

$$\begin{aligned} (\sigma^{2d-1,p} * \sigma^{2d-1,p})^G &= \{(x, \frac{1}{2}, \tau x) \in \sigma^{2d-1,p} * \sigma^{2d-1,p} | \ x \in \sigma^{2d-1,p}\} \\ &\in \mathcal{S}^{2d-1,2d-1} \\ (\Sigma^{1,0}\sigma^{2d-1,p})^G &= \{(\frac{1}{2}, x) \in \Sigma^{1,0}\sigma^{2d-1,p} | \ x \in (\sigma^{2d-1,p})^G\} \\ &\in \mathcal{S}^{p,p}. \end{aligned}$$

Thus we have that

$$\begin{aligned} \sigma^{2d-1,p} * \sigma^{2d-1,p} &\in \mathcal{S}^{4d-1,2d-1} \\ \Sigma^{1,0}\sigma^{2d-1,p} &\in \mathcal{S}^{2d,p}. \end{aligned}$$

Hence theorem 5.10 says

$$(2d-1,p) = (4d-1,2d) \text{ or } (2d-1,d) \text{ or } (2d-1,0).$$

The solutions without contradictions are $p = d, 0$.

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