Homotopy-commutativity
in rotation groups

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1 Introduction

Assume $G$ is a topological group and $S$, $S'$ are subspaces of $G$, each of which contains the unit as its base point. There is the commutator map $c$ from $S \wedge S'$ to $G$ which maps $(x, y) \in S \wedge S'$ to $xyx^{-1}y^{-1} \in G$. We say $S$ and $S'$ homotopy-commute in $G$ if $c$ is null homotopic.

In this paper, we describe the homotopy-commutativity of the case $G = SO(n + m - 1)$, $S = SO(n)$ and $S' = SO(m)$ where $n, m > 1$. Here we use the usual embeddings

$$SO(1) \subset SO(2) \subset SO(3) \subset \cdots.$$ 

Trivially $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n + m)$. And it is known that if $n + m > 4$, $SO(n)$ and $SO(m)$ do not homotopy-commute in $SO(n + m - 2)$. (See [1] and [2].) But the homotopy-commutativity in $SO(n + m - 1)$ has not been solved exactly.

We shall say a pair $(n, m)$ is irregular if $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n + m - 1)$, and regular if they do not. In [1] the following problem is proposed: "when is $(n, m)$ irregular?", and the next theorem is showed.

(James and Thomas) Let $n + m \neq 4, 8$. If $n$ or $m$ is even or if $d(n) = d(m)$ then $(n, m)$ is regular, where $d(q)$, for $q \geq 2$, denotes the greatest power of 2 which divides $q - 1$. 

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In this paper we shall prove the more strict result as showed in the next theorem.

If \( n \) or \( m \) is even or if \( (n + m - 2) \equiv 0 \) mod 2 then \((n, m)\) is regular.

We identify \( \text{RP}^{k-1} \to \text{SO}(k) \) by the following way. Let \( i'_k : \text{RP}^{k-1} \to \text{O}(k) \) be the map which attaches a line \( l \in \text{RP}^{k-1} \) with \( i'_k(l) \in \text{O}(k) \) defined by
\[
  i'_k(l)(v) = v - 2(v, e)e,
\]
where \( e \) is a unit vector of \( l \) and \( v \in \mathbb{R}^k \). And let \( i_k(l) = i'_k(l_0)^{-1} \cdot i'_k(l) \) where \( l_0 \) is the base point of \( \text{RP}^{k-1} \). Then \( i_k \) preserves the base points.

Theorem 1.2 follows from the next theorem.

Let \( n \) and \( m \) be odd. \( \text{RP}^{n-1} \subset \text{SO}(n) \) and \( \text{RP}^{m-1} \subset \text{SO}(m) \) homotopy-commute in \( \text{SO}(n + m - 1) \) if and only if \( (n + m - 2) \equiv 1 \) mod 2.

Let \( \text{SO} \) be \( \lim_{\to} (\text{SO}(1) \subset \text{SO}(2) \subset \text{SO}(3) \subset \cdots) \) and consider the fibration \( \text{SO}(n + m - 1) \to \text{SO} \to \text{SO}/\text{SO}(n + m - 1) \). Then we have a sequence of spaces
\[
\cdots \to \Omega \text{SO} \xrightarrow{\delta} \Omega(\text{SO}/\text{SO}(n + m - 1)) \xrightarrow{i} \text{SO} \xrightarrow{p} \text{SO}/\text{SO}(n + m - 1).
\]

We can see \( i \circ c \mid \text{RP}^{n-1} \wedge \text{RP}^{m-1} \simeq * : \text{RP}^{n-1} \wedge \text{RP}^{m-1} \to \text{SO} \). This means there exists \( \lambda : \text{RP}^{n-1} \wedge \text{RP}^{m-1} \to \Omega\text{SO}/\text{SO}(n + m - 1) \) such that \( \delta \circ \lambda = c \mid \text{RP}^{n-1} \wedge \text{RP}^{m-1} \). The construction of \( \lambda \) and the cohomology map \( \lambda^* \) are argued in §3. We describe about lifts of \( \lambda \) in §3 and finally, in §4, we determine when a lift of \( \lambda \) exists, which means when \( c \mid \text{RP}^{n-1} \wedge \text{RP}^{m-1} \simeq * \).

2 Lift \( \lambda \) of \( c \)

Definition A sequence of spaces \( X_i \) and continuous maps \( f_i \)
\[
\cdots \xrightarrow{f_i} X_i \xrightarrow{f_i} \cdots \xrightarrow{f_0} X_0
\]
is called a fibration sequence if, for any \( i \geq 0 \), there exists a fibration \( Y_i^{(2)} \xrightarrow{i_i} Y_i^{(1)} \xrightarrow{\pi_i} Y_i^{(0)} \), homotopy equivalence maps \( \psi_i^{(k)} : X_{i+k} \to Y_i^{(k)} \) (\( k = 0, 1, 2 \)), and
the following diagram commutes up to homotopy.

\[
\begin{array}{ccc}
X_{i+2} & \xrightarrow{f_{i+1}} & X_{i+1} \\
\simeq \downarrow \psi_{i}^{(2)} & & \simeq \downarrow \psi_{i}^{(1)} \\
Y_{i}^{(2)} & \xrightarrow{j_{i}} & Y_{i}^{(1)} \\
\end{array}
\]

For example, given a fibration \( F \to E \to B \), there is a fibration sequence

\[
\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B.
\]

Consider the fibration \( SO \to SO/\text{SO}(n + m - 1) \) with the fibre \( \text{SO}(n + m - 1) \).

Then we have a fibration sequence.

\[
\cdots \to \Omega SO \xrightarrow{\Omega \pi} \Omega(SO/\text{SO}(n + m - 1)) \xrightarrow{\delta} \text{SO}(n + m - 1) \xrightarrow{i} \text{SO} \xrightarrow{p} SO/\text{SO}(n + m - 1)
\]

Obviously \( i \circ c : SO(n) \wedge SO(m) \to SO \) is null homotopic. This means there exists a lift of \( c \), that is, a map \( \lambda : \text{RP}^{n-1} \wedge \text{RP}^{m-1} \to \Omega(SO/\text{SO}(n + m - 1)) \)
such that \( \delta \circ \lambda \simeq c \).

In R.Bott\[3\] it is showed that the following map \( \lambda_{0} : SO(n) \wedge SO(m) \to \Omega(SO/\text{SO}(n + m - 1)) \) is a lift of \( c \).

Recall the fibration \( SO(k - 1) \to SO(k) \xrightarrow{p_{k}} S^{k-1} \). Define \( h \) as \( h = \Sigma(p_{n} \wedge p_{m}) : \Sigma(SO(n) \wedge SO(m)) \to \Sigma(S^{n-1} \wedge S^{m-1}) \simeq S^{n+m-1} \).

Then \( \text{ad}h \) is a lift of \( c \) in the following fibration sequence. (See [5].)

\[
\cdots \to \Omega SO(n + m) \to \Omega S^{n+m-1} \to SO(n + m - 1) \to SO(n + m) \to S^{n+m-1}
\]

The fibration \( SO(n + m) \to S^{n+m-1} \) is the restriction of \( SO \to SO/\text{SO}(n + m - 1) \) to \( S^{n+m-1} = SO(n + m)/\text{SO}(n + m - 1) \xrightarrow{\delta} SO/\text{SO}(n + m - 1) \).

Therefore we define \( \lambda_{0} \) as \( \Omega j \circ \text{ad}h \). Refer to the commutative diagram below.

The rest of this section is devoted to the computation of the cohomology map of \( \lambda \). And throughout this paper we use \( \mathbb{Z}/2\mathbb{Z} \) as the coefficient ring of cohomology unless mentioned.

First we refer to the cohomology rings of spaces which are used in this paper, that is,

\[
\begin{align*}
\text{H}^{*}(\Omega_{0} \text{SO}) & = \mathbb{Z}/2\mathbb{Z}[\alpha_{2}, \alpha_{4}, \alpha_{6}, \cdots]/(\alpha_{4k} - \alpha_{2k}^{2}), \\
\text{H}^{*}(\Omega(\text{SO}/\text{SO}(n + m - 1))) & = \mathbb{Z}/2\mathbb{Z}[\alpha'_{n+m-2}, \alpha'_{n+m}, \cdots]/(\alpha'_{4k} - \alpha_{2k}^{2}), \\
\text{H}^{*}(\text{SO}(k)) & = \Delta(x_{1}, \cdots, x_{k-1}), \\
\text{H}^{*}(\text{SO}(k)/\text{SO}(k-l)) & = \Delta(x'_{k-l}, \cdots, x'_{k-1}),
\end{align*}
\]

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where deg($\alpha_{2i}$) = 2$i$, deg($\alpha'_{2i}$) = 2$i$, deg($x_i$) = $i$. And also

$$\Omega p^*(\alpha'_k) = \alpha_k.$$ 

$\lambda_0^*(\alpha'_{n+m-2}) = x_{n-1} \otimes x_{m-1}$. Proof. Consider the fibration $p_k : SO(k) \rightarrow S^{k-1}$ with the fibre $SO(k-1)$. Let $c_i$ be the generator of $H^i(S^k)$. Then $p_k^*(c_{k-1}) = x_{k-1}$. Thus we have

$$h^*(c_{n+m-1}) = \Sigma(p_n \wedge p_m)^*(\Sigma c_{n-1} \otimes c_{m-1}) = \Sigma(x_{n-1} \otimes x_{m-1}).$$

Hence $(\text{ad}h)^*(\sigma c_{n+m-1}) = x_{n-1} \otimes x_{m-1}$, where $\sigma$ is the cohomology suspension $\sigma : H^{*+1}(X) \rightarrow H^*(\Omega X)$. On the other hand, $j^*(x_{n+m-1}) = c_{n+m-1}$ means

$$(\Omega j)^*(\alpha'_{n+m-2}) = (\Omega j)^*(\sigma x'_{n+m-1}) = \sigma c_{n+m-1}.$$ 

Therefore it follows that

$$\lambda_0^*(\alpha'_{n+m-2}) = (\text{ad}h)^*(\Omega j)^*(\alpha'_{n+m-2}) = x_{n-1} \otimes x_{m-1}.$$ 

Q.E.D.

Now let $\lambda = \lambda_0 \circ (i_m \wedge i_n) : \mathbb{R}P^{n-1} \wedge \mathbb{R}P^{m-1} \rightarrow \Omega(SO/SO(n+m-1))$ and in the following we use $c$ as the commutator map from $\mathbb{R}P^{n-1} \wedge \mathbb{R}P^{m-1}$ to $SO(n+m-1)$. Easily we have $i_k^*(x_{k-1}) = \tau^{k-1}$ where $\tau$ means the generator of $H^1(\mathbb{R}P^{k-1})$. (See Whitehead [4].) Thus

$$\lambda^*(\alpha'_{n+m-2}) = (i_m \wedge i_n)^* \circ \lambda_0^*(\alpha'_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}.$$ 

### 3 Lift of $\lambda$ and homotopy commutativity

In this section we prove the next theorem.

Let $n, m$ be odd.
1. \( c \simeq * \) if and only if there exists a lift of \( \lambda \), that is, a map \( x : \mathbb{R}P^{n-1} \land \mathbb{R}P^{m-1} \to \Omega_0(\text{SO}) \) such that \( \lambda = \Omega p \circ x \).

2. \( c \simeq * \) if and only if there exists \( x : \mathbb{R}P^{n-1} \land \mathbb{R}P^{m-1} \to \Omega_0(\text{SO}) \) such that \( x^*(\alpha_{n+m-2}) \simeq \tau^{n-1} \otimes \tau^{m-1} \).

**Proof.**

1. The sequence

\[
\cdots \to \Omega_0(\text{SO}) \frac{\pi_0}{\Omega(\text{SO}/\text{SO}(n+m-1))} \frac{\delta}{\to} \text{SO}(n+m-1)
\]

is a fibration sequence and \( \lambda \) is a lift of \( c \). Therefore the statement follows.

2. By the first statement it is sufficient to prove that \( x \) is a lift of \( \lambda \) if and only if \( x^*(\alpha_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1} \). We need the following lemma.

Let \( n \) and \( m \) be odd. Then

\[
\pi_i(\text{SO}/\text{SO}(n+m-1)) = \begin{cases} 0 & i \leq n + m - 2 \\ \mathbb{Z}/2\mathbb{Z} & i = n + m - 1 \end{cases}
\]

**Proof.** Consider the fibration

\[\text{SO}(n+m+1)/\text{SO}(n+m-1) \to \text{SO}/\text{SO}(n+m-1) \to \text{SO}/\text{SO}(n+m+1)\]

and see the homotopy exact sequence. Remark that \( \pi_i(\text{SO}/\text{SO}(2k+1)) = 0 \) for \( i \leq 2k \) and we obtain

\[\pi_{n+m-1}(\text{SO}/\text{SO}(n+m-1)) = \pi_{n+m-1}(\text{SO}(n+m+1)/\text{SO}(n+m-1)).\]

It is known that \( \pi_{n+m-1}(\text{SO}(n+m+1)/\text{SO}(n+m-1)) = \mathbb{Z}/2\mathbb{Z} \) provided \( n + m - 1 \) is odd. Hence we obtained the statement.

Q.E.D.

By Lemma 3.6 it follows that

\[
\pi_i(\Omega(\text{SO}/\text{SO}(n+m-1))) = \begin{cases} 0 & i \leq n + m - 3 \\ \mathbb{Z}/2\mathbb{Z} & i = n + m - 2. \end{cases}
\]

Now add cells \( e_i(i \geq 1) \) to \( \Omega(\text{SO}/\text{SO}(n+m-1)) \) so that \( \pi_k(\Omega(\text{SO}/\text{SO}(n+m-1)) \) vanishes for \( k \geq n + m - 1 \), where \( \text{dim} e_i \geq n + m \). We shall call the obtained space \( K \), that is,

\[
\Omega(\text{SO}/\text{SO}(n+m-1)) \hookrightarrow \Omega(\text{SO}/\text{SO}(n+m-1)) \cup e_1 \cup e_2 \cup \cdots = K \quad (1)
\]
and
\[ \pi_i(K) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = n + m - 2 \\ 0 & \text{otherwise.} \end{cases} \]
(2)
Thus \( K \) is an Eilenberg-Maclane space \( K(\mathbb{Z}/2\mathbb{Z}; n + m - 2) \). Let \( \gamma \) denote the inclusion map from \( \Omega(\text{SO}/\text{SO}(n + m - 1)) \) to \( K \). Here
\[ \gamma_* : \pi_{n+m-2}(\Omega(\text{SO}/\text{SO}(n + m - 1))) \to \pi_{n+m-2}(K) \]
is not a 0-map. This means that by the isomorphism
\[ [\Omega(\text{SO}/\text{SO}(n + m - 1)), K] \cong H^{n+m-2}(\Omega(\text{SO}/\text{SO}(n + m - 1))) \]
\( \gamma \) corresponds to \( \alpha'_{n+m-2} \), that is, \( \gamma^* u = \alpha'_{n+m-2} \) where \( u \) is the generator of \( H^{n+m-2}(K) \).

On the other hand, (1) and (2) imply that \( \gamma_* : \pi_i(\Omega(\text{SO}/\text{SO}(n + m - 1))) \to \pi_i(K) \) is isomorphic for \( i \leq n + m - 2 \) and epic for \( i \geq n + m - 1 \). Then by Whitehead’s theorem
\[ [\text{RP}^{n-1} \wedge \text{RP}^{m-1}, \Omega(\text{SO}/\text{SO}(n + m - 1))] \cong [\text{RP}^{n-1} \wedge \text{RP}^{m-1}, K] \cong H^{n+m-2}(\text{RP}^{n-1} \wedge \text{RP}^{m-1}). \]
Thus maps \( f \) and \( g : \text{RP}^{n-1} \wedge \text{RP}^{m-1} \to \Omega(\text{SO}/\text{SO}(n + m - 2)) \) are homotopic if and only if \( f^*(\alpha'_{n+m-2}) = g^*(\alpha'_{n+m-2}) \).

Now we assume \( x : \text{RP}^{n-1} \wedge \text{RP}^{m-1} \to \Omega(\text{SO}/\text{SO}(n + m - 1)) \) satisfies that \( x^*(\alpha_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1} \). Then
\[ (\Omega p \circ x)^*(\alpha'_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}. \]
By §2 \( \lambda^*(\alpha'_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1} \). Thus we obtain \( \Omega p \circ x \simeq \lambda \) and \( x \) is a lift of \( \lambda \).

The inverse is trivial and the proof of theorem 3.5 is finished.

4 Existence of lift of \( \lambda \)

In this section we prove the next theorem which completes the proof of Theorem 1.3. Let \( n \) and \( m \) be odd. There exists a map \( x : \text{RP}^{n-1} \wedge \text{RP}^{m-1} \to \Omega_0(\text{SO}) \) such that \( x^*(\alpha_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1} \) if and only if
\[ \binom{n+m-2}{n-1} \equiv 1 \mod 2. \]
Then the total Stiefel Whitney class to the $\pi$ projection and decompose generator of $H$ and let $s$ over $r$

Proof. By direct computation, we see

$$\theta := (r_1 - 1) \otimes (r_1 - 1) \otimes (r_\infty - 1) \otimes (r_\infty - 1) \in \widehat{KO}(\Sigma^2(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty)).$$

Here $r_1$ is the Möbius line bundle over $S^1$ and $r_\infty$ is the canonical line bundle over $\mathbb{RP}^\infty$. Now we compute the total Stiefel Whitney class of $\theta$. We start from the next lemma.

Let $A = 1 + a_1 + a_2 + \cdots \in H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty)$ where $a_i \in H^i(\mathbb{RP}^\infty \times \mathbb{RP}^\infty)$ and let $s_i \in H^i(S^1 \times S^1 \times \mathbb{RP}^\infty \times \mathbb{RP}^\infty)$ ($i = 1, 2$) be the pull back of the generator of $H^1(S^1)$ by the canonical projection from $S^1 \times S^1 \times \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ to the $i$th factor $S^1$. Then we have

$$\frac{(A + s_1 + s_2)A}{(A + s_1)(A + s_2)} = \frac{A^2 + s_1s_2}{A^2} \in H^*(S^1 \times S^1 \times \mathbb{RP}^\infty \times \mathbb{RP}^\infty).$$

Proof. By direct computation, we see

$$\frac{(A + s_1 + s_2)A}{(A + s_1)(A + s_2)} = \frac{((A + s_1)^2 + (A + s_1)s_2)A}{(A + s_1)^2(A + s_2)} = \frac{(A^2 + s_2A + s_1s_2)A}{A^2(A + s_2)} = \frac{A(A + s_2)^2 + (A + s_2)s_1s_2}{A(A + s_2)^2} = \frac{A^2 + s_1s_2}{A^2}.$$

Q.E.D.

Let $\pi : S^1 \times S^1 \times \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \Sigma^2(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty)$ be the canonical projection and decompose $\pi^*\theta$ as

$$\pi^*\theta = r_1 \times r_1 \times r_\infty \times r_\infty + 1 \times 1 \times r_\infty \times r_\infty - 1 \times r_1 \times r_\infty \times r_\infty - r_1 \times 1 \times r_\infty \times r_\infty$$

$$- r_1 \times 1 \times r_\infty - 1 \times 1 \times r_\infty + 1 \times r_1 \times r_\infty + r_1 \times 1 \times r_\infty$$

$$- r_1 \times r_1 \times r_\infty \times 1 - 1 \times 1 \times r_\infty \times 1 + 1 \times r_1 \times r_\infty \times 1 + r_1 \times 1 \times r_\infty \times 1$$

$$+ r_1 \times r_1 \times 1 \times 1 + 1 \times 1 \times 1 - 1 \times r_1 \times 1 + 1 \times 1 \times 1.$$ 

Then the total Stiefel Whitney class $w(\pi^*\theta)$ of $\pi^*\theta$ is given by

$$w(\pi^*\theta) = \frac{(1 + \tau_1 + \tau_2 + s_1 + s_2)(1 + \tau_1 + \tau_2)(1 + s_1 + s_2)}{(1 + \tau_1 + \tau_2 + s_1)(1 + \tau_1 + \tau_2 + s_2)(1 + s_1)(1 + s_2)} \cdot \left(\frac{(1 + \tau_1 + \tau_2 + s_1 + s_2)(1 + \tau_1)}{(1 + \tau_1 + s_1)(1 + \tau_1 + s_2)} \cdot \frac{(1 + \tau_2 + s_1 + s_2)(1 + \tau_2)}{(1 + \tau_2 + s_1)(1 + \tau_2 + s_2)}\right)^{-1}. $$

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Here $\tau_i$ ($i = 1, 2$) is the pull back of the generator of the cohomology ring of the $i$th factor of $\mathbb{RP}^\infty \times \mathbb{RP}^\infty$. By the previous lemma, we obtain

$$w(\pi^*\theta) = \frac{1 + \tau_1^2 + \tau_2^2 + s_1 s_2}{1 + \tau_1^2 + \tau_2^2} \cdot (1 + s_1 s_2) \cdot \left( \frac{1 + \tau_1^2 + s_1 s_2}{1 + \tau_1^2} \right)^{1 - 1} \cdot \left( \frac{1 + \tau_2^2 + s_1 s_2}{1 + \tau_2^2} \right)^{-1}$$

$$= \{ 1 + (1 + \tau_1^2 + \tau_2^2)^{-1} s_1 s_2 \} (1 + s_1 s_2) \{ 1 + (1 + \tau_1^2)^{-1} s_1 s_2 \} \{ 1 + (1 + \tau_2^2)^{-1} s_1 s_2 \}$$

$$= 1 + s_1 s_2 \{ (1 + \tau_1^2 + \tau_2^2)^{-1} + 1 + (1 + \tau_1^2)^{-1} + (1 + \tau_2^2)^{-1} \}$$

$$= 1 + s_1 s_2 \left\{ \sum_{i=0}^\infty (\tau_1^2 + \tau_2^2)^i + 1 + \sum_{i=0}^\infty \tau_1^{2i} + \sum_{i=0}^\infty \tau_2^{2i} \right\}$$

Therefore we see

$$w(\theta) = 1 + \sum_{i=0}^\infty \sum_{j=1}^{i-1} \left( i \right) \tau_1^{2j} \tau_2^{2i-2j}.$$ 

Let $f$ be the classifying map of $\theta$, that is, the map

$$f : \Sigma^2(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty) \to \text{BSO}$$

such that $f^*(\xi) = \theta$ where $\xi = \lim_{n \to -\infty} (\xi_n - n)$ and $\xi_n$ is the universal $SO(n)$ vector bundle over $\text{BSO}(n)$.

It is known that $H^*(\text{BSO}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \cdots]$ where $w_i$ is the $i$th Stiefel Whitney class. Let $\iota_k : \mathbb{RP}^k \to \mathbb{RP}^\infty$ be the inclusion map and let

$$x_0 := (\text{ad}^2 f) \circ (\iota_{n-1} \wedge \iota_{m-1}) : \mathbb{RP}^{n-1} \wedge \mathbb{RP}^{m-1} \to \Omega\text{SO}.$$ 

Then it follows that for $N \geq 1$

$$x_0^*(\alpha_{2N}) = (\iota_{n-1} \wedge \iota_{m-1})^* (\text{ad}^2 f)^* \sigma^2 w_{2N+2}$$

$$= (\iota_{n-1} \wedge \iota_{m-1})^* \left( \sum_{j=1}^{N-1} \binom{2N}{2j} \tau_2^{2j} \wedge \tau_2^{2N-2j} \right).$$

Particularly $x_0^*(\alpha_{n+m-2}) = \binom{n+m-2}{n-1} \tau^{n-1} \wedge \tau^{m-1}$. Thus if $\binom{n+m-2}{n-1} = 1$ then there exists $x_0 : \mathbb{RP}^{n-1} \wedge \mathbb{RP}^{m-1} \to \Omega\text{SO}$ such that $x_0^*(\alpha_{n+m-2}) = \tau^{n-1} \wedge \tau^{m-1}$.
Now we shall prove the inverse, that is, prove that if \( \binom{n+m-2}{n-1} \equiv 0 \mod 2 \) then \( x^*(\alpha_{n+m-2}) = 0 \) for any \( x : \mathbb{R}P^{n-1} \land \mathbb{R}P^{m-1} \rightarrow \Omega \mathbb{SO} \). Let \( n = 2a + 1, m = 2b + 1 \) where \( a, b \in \mathbb{Z}, a, b \geq 1 \). Moreover we set \( a \leq b \).

Here we use the Steenrod’s square operators \( Sq^i \). In \( H^*(\Omega_0 \mathbb{SO}) \), \( Sq^i \) acts as follows

\[
Sq^i(\alpha_{2j}) = \begin{cases} 
    \binom{2j+1}{i}\alpha_{2j+i} & \text{if } i \text{ is even} \\
    0 & \text{if } i \text{ is odd.}
\end{cases}
\]

Let \( x : \mathbb{R}P^{2a} \land \mathbb{R}P^{2b} \rightarrow \Omega_0 \mathbb{SO} \) be an arbitrary map.

We set \( a, b, x \) as above then

\[
x^*(\alpha_2) = 0 \text{ and } x^*(\alpha_6) = \tau^2 \otimes \tau^4 + \tau^4 \otimes \tau^2 \text{ or 0.}
\]

**Proof.** Since \( x^*(\alpha_2) \in H^*(\mathbb{R}P^{2a} \land \mathbb{R}P^{2b}) \), \( x^*(\alpha_2) = \tau \otimes \tau \) or 0. If \( x^*(\alpha_2) = \tau \otimes \tau \), then we have

\[
Sq^1 x^*(\alpha_2) = \tau^2 \otimes \tau + \tau \otimes \tau^2.
\]

On the other hand,

\[
Sq^1 x^*(\alpha_2) = x^*(Sq^1 \alpha_2) = 0.
\]

Therefore \( x^*(\alpha_2) = 0 \).

Next we consider \( x^*(\alpha_6) \). If \( (a, b) = (1, 1) \) then \( x^*(\alpha_6) = 0 \), and if \( (a, b) = (1, 2) \) we can see \( x^*(\alpha_6) = \tau^2 \otimes \tau^4 \) or 0 as asserted. And otherwise, set

\[
x^*(\alpha_6) = \rho_1 \tau \otimes \tau^5 + \rho_2 \tau^2 \otimes \tau^4 + \rho_3 \tau^3 \otimes \tau^3 + \rho_4 \tau^4 \otimes \tau^2 + \rho_5 \tau^5 \otimes \tau^1,
\]

where \( \rho_i \in \mathbb{Z}/2\mathbb{Z} \) and the statement follows the next two equations.

\[
Sq^1 x^*(\alpha_6) = x^*(Sq^1 \alpha_6) = 0
\]

\[
Sq^2 x^*(\alpha_6) = x^*(\alpha_8) = x^*(\alpha_2)^4 = 0
\]

Q.E.D.

Remark that if \( 2(a+b) = 2d - 2 \) for some \( d \in \mathbb{N} \), then \( \binom{2(a+b)}{2i} \equiv 1 \mod 2 \) for any \( i \in \mathbb{Z} \) such that \( 0 \leq i \leq a + b \). And also when \( 2(a+b) = 2d \) for some \( d \in \mathbb{N} \),

\[
\binom{2(a+b)}{2i} \equiv \begin{cases} 
    1 \mod 2 & \text{if } i = 0 \text{ or } a + b \\
    0 \mod 2 & \text{otherwise.}
\end{cases}
\]
In this case
\[ x^*(\alpha_{2(a+b)}) = x^*(\alpha_2) = x^*(\text{a power of } \alpha_2) = 0 \]
as asserted. Hence we can assume that \(2(a+b) \neq 2^k\) or \(2^k - 2\) for any \(k \in \mathbb{N}\).

Next we shall prove the next theorem. Let \(a, b\) and \(x\) be as above. If \(x^*(\alpha_6) = 0\) then \(x^*(\alpha_{2(a+b)}) = 0\). \textit{Proof.} Let \(d\) be the number which satisfies
\[ 2^d < 2(a+b) < 2^{d+1} - 2 \quad d \in \mathbb{N} \quad (d \geq 3) \]
We distinguish between the following two cases.

I) \[ 2^d < 2(a+b) < 3 \cdot 2^{d-1} - 2 \quad (3) \]

II) \[ 3 \cdot 2^{d-1} - 2 \leq 2(a+b) < 2^{d+1} - 2 \quad (4) \]

Let \(a, b\) and \(x\) be as above. In any of the case I) and II), if \(x^*(\alpha_6) = 0\) then one of the following holds.

i) \(x^*(\alpha_{2k-2}) = 0\) for \(3 \leq k \leq d - 1\).

ii) \(2a = 2^r - 2\) for some \(r \in \mathbb{N}\), \(r \leq d - 1\) and
\[
x^*(\alpha_{2^r-2}) = \begin{cases} 
0 & \text{for } 3 \leq k \leq r \\
\tau^{2^r-2} \otimes \tau^{2^{k-1}} & \text{for } r + 1 \leq k \leq d - 1.
\end{cases}
\]

\textit{Proof.} We use induction, that is, we prove the next two propositions.

a) If \(x^*(\alpha_{2^k-1-2}) = 0\) and \(4 \leq k \leq d - 1\), then one of the followings holds.
\[
\begin{align*}
\bullet & \quad x^*(\alpha_{2^k-2}) = 0. \\
\bullet & \quad 2a = 2^{k-1} - 2 \text{ and } x^*(\alpha_{2^k-2}) = \tau^{2^{k-1}-2} \otimes \tau^{2^{k-1}}.
\end{align*}
\]

b) If \(2a = 2^r - 2\) and \(x^*(\alpha_{2^k-1-2}) = \tau^{2r-2} \otimes \tau^{2^{k-1}-2r}\) and \(r + 2 \leq k \leq d - 1\), then
\[ x^*(\alpha_{2^k-2}) = \tau^{2r-2} \otimes \tau^{2^{k-1}-2r}. \]
First we assume \(4 \leq k \leq d - 1\) and \(x^* (\alpha_{2^{k-1}}) = 0\) and prove a). Let

\[
x^* (\alpha_{2^k - 2}) = \sum_{i=s}^{t} \rho_i \tau^i \otimes \tau^{(2^k-2)-i},
\]

where

\[
\begin{align*}
s &= \max \{1, (2^k - 2) - 2b\}, \\
t &= \min \{2^k - 3, 2a\}, \\
\rho_i &\in \mathbb{Z}/2\mathbb{Z}.
\end{align*}
\]

Since \(\text{Sq}^1 (x^* (\alpha_{2^k - 2})) = x^* (\text{Sq}^1 \alpha_{2^k - 2}) = 0\), we have that

\[
\text{Sq}^1 \left( \sum_{i=s}^{t} \rho_i \tau^i \otimes \tau^{(2^k-2)-i} \right) = 0.
\]

Here, \(\tau^i \otimes \tau^{(2^k-2)-i+1} \neq 0\) for \(s + 1 \leq i \leq t\). Therefore

\[
\rho_i = 0 \quad \text{for } i : \text{ odd}, \; s \leq i \leq t. \tag{5}
\]

Next we use \(\text{Sq}^2\). By (5) we can set

\[
x^* (\alpha_{2^k - 2}) = \sum_{i=s'}^{t'} \rho_{2i} \tau^{2i} \otimes \tau^{(2^k-2)-2i},
\]

where

\[
\begin{align*}
s' &= \max \{1, \frac{2^k - 2}{2} - b\} \\
t' &= \min \left\{ \frac{2^k - 4}{2}, a \right\}.
\end{align*}
\]

Since

\[
\begin{align*}
\text{Sq}^2 x^* (\alpha_{2^k - 2}) &= x^* (\text{Sq}^2 \alpha_{2^k - 2}) \\
&= x^* (\alpha_{2^k}) \\
&= x^* (\alpha_{2^{k-1}}) \\
&= 0,
\end{align*}
\]

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we have

\[ \text{Sq}^2 \left( \sum_{i=s'}^{t'} \rho_{2i} \tau^{2i} \otimes \tau^{(2^k-2)-2i} \right) \]

\[ = \sum_{s' \leq 2j \leq t'} \rho_{4j} \text{Sq}^2(\tau^{4j} \otimes \tau^{(2^k-2)-4j}) + \sum_{s' \leq 2j-1 \leq t'} \rho_{4j-2} \text{Sq}^2(\tau^{4j-2} \otimes \tau^{(2^k-2)-4j+2}) \]

\[ = \sum_{s' \leq 2j \leq t'} \rho_{4j} \tau^{4j} \otimes \tau^{2^k-4j} + \sum_{s' \leq 2j-1 \leq t'} \rho_{4j-2} \tau^{4j} \otimes \tau^{2^k-4j} = 0 \quad (6) \]

Here \( \tau^l \otimes \tau^{2^k-l} \neq 0 \) for \( s' + 1 \leq 2j \leq t' \). Thus

\[ \rho_{4j} = \rho_{4j-2} \text{ for } s' + 1 \leq 2j \leq t'. \quad (7) \]

Next we consider \( \text{Sq}^4 \). Since

\[ \text{Sq}^4 x^*(\alpha_{2^k-2}) = x^*(\text{Sq}^4 \alpha_{2^k-2}) \]

\[ = x^*(\alpha_{2^k+2}) \]

\[ = x^*(\text{Sq}^{2^k-1} \text{Sq}^4 \alpha_{2^k-1-2}) \]

\[ = \text{Sq}^{2^k-1} \text{Sq}^4 x^*(\alpha_{2^k-1-2}) \]

\[ = 0, \]

we have that

\[ \text{Sq}^4 \left( \sum_{i=s'}^{t'} \rho_{2i} \tau^{2i} \otimes \tau^{(2^k-2)-2i} \right) \]

\[ = \text{Sq}^4 \left( \sum_{s' \leq 4j \leq t'} \rho_{8j} \tau^{8j} \otimes \tau^{(2^k-2)-8j} + \sum_{s' \leq 4j-1 \leq t'} \rho_{8j-2} \tau^{8j-2} \otimes \tau^{(2^k-2)-8j+2} \right) \]

\[ + \sum_{s' \leq 4j-2 \leq t'} \rho_{8j-4} \tau^{8j-4} \otimes \tau^{(2^k-2)-8j+4} + \sum_{s' \leq 4j+1 \leq t'} \rho_{8j+2} \tau^{8j+2} \otimes \tau^{(2^k-2)-8j-2} \]

\[ = \sum_{s' \leq 4j \leq t'} \rho_{8j} \tau^{8j} \otimes \tau^{2^k+2-8j} + \sum_{s' \leq 4j-1 \leq t'} \rho_{8j-2} \tau^{8j-2} \otimes \tau^{2^k-8j} \]

\[ + \sum_{s' \leq 4j-2 \leq t'} \rho_{8j-4} \tau^{8j-4} \otimes \tau^{2^k+2-8j} + \sum_{s' \leq 4j+1 \leq t'} \rho_{8j+2} \tau^{8j+2} \otimes \tau^{2^k-8j} \]

\[ = 0. \quad (8) \]
Thus

\[
\begin{cases}
\rho_{8j} = \rho_{8j-4} \text{ for } s' + 2 \leq 4j \leq t' \\
\rho_{8j-2} = \rho_{8j+2} \text{ for } s' + 1 \leq 4j \leq t' - 1
\end{cases}
\]  

(9)

We set \( A \) as the set \( \{ i \in \mathbb{N} \mid s' \leq i \leq t' \} \). (7) and (9) mean that

\[
2i, 2i - 1 \in A \quad \text{then} \quad \rho_{4i-2} = \rho_{4i};
\]

(10)

\[
4i, 4i - 2 \in A \quad \text{then} \quad \rho_{8i} = \rho_{8i-4};
\]

(11)

\[
4i - 1, 4i + 1 \in A \quad \text{then} \quad \rho_{8i-2} = \rho_{8i+2}.
\]

(12)

Therefore, for \( i \in A - \{s', t' - 1, t'\} \), \( \rho_{2i} = \rho_{2i+2} \). The reason is this: if \( i \) is odd, it is trivial from (10); if \( i = 4j \) for some \( j \), \( \rho_{8j} = \rho_{8j-2} = \rho_{8j+2} \); if \( i = 4j - 2 \) for some \( j \), \( \rho_{8j-4} = \rho_{8j} = \rho_{8j-2} \).

We obtain that

\[
\rho_{2s'+2} = \rho_{2s'+4} = \cdots = \rho_{2t'-2}.
\]

Also, we see

\[
2b \geq a + b > 2^{d-1} \text{ and } \frac{2^k - 2}{2} - b \leq \frac{2^{d-1} - 2}{2} - 2^{d-2} < 1
\]

(13)

and we have

\[
s' = \max\{1, \frac{2^k - 2}{2} - b\} = 1.
\]

We see again (8) and look into the term of \( \tau^2 \otimes \tau^{2k} \), then we have that \( \rho_2 = 0 \) and from (10) \( \rho_2 = \rho_4 \). Hence we have

\[
0 = \rho_2 = \rho_4 = \cdots = \rho_{2t'-2},
\]

that is,

\[
x^*(\alpha_{2^{k-2}}) = \rho_{2t'} \tau^{2t'} \otimes \tau^{(2^k-2)-2t'}.
\]

(14)

If \( 2a \geq 2^k - 4 \) then we have

\[
t' = \min\{\frac{2^k - 4}{2}, a\} = 2^{k-1} - 2
\]

and from (10)

\[
\rho_{2t'-2} = \rho_{2t'},
\]

that is,

\[
x^*(\alpha_{2^{k-2}}) = 0.
\]

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Therefore we can assume
\[ 2a < 2^k - 4, \]
that is, \( t' = a \). Here if \( 2a = 2^{k-1} - 2 \), then by (14) \( x^*(\alpha_{2^k-2}) = \tau^{2k-2} \otimes \tau^{2k-1} \) or 0 as asserted. Hence what we have to prove prove is that if \( 2a \neq 2^{k-1} - 2 \) then \( \rho_{2^{k-1}} = 0 \).

We set \( p(2a) \) so that \( 2^{p(2a)} \) is the greatest power of 2 which devides \( 2a + 2 \).

Let \( p := p(2a) \). We remark that \( p \leq k - 2 \) since, if it were not, by (15) \( 2a = 2^{k-1} - 2 \). Using \( \text{Sq}^{2^p} \), we see
\[
\text{Sq}^{2^p} x^*(\alpha_{2^k-2}) = x^*(\alpha_{2^k+2^p-2}) = \text{Sq}^{2^{k-1}} \text{Sq}^{2^p} x^*(\alpha_{2^k-1-2}) = 0.
\]

Thus it follows that
\[
\text{Sq}^{2^p} (\rho_{2^d} \tau^{2a} \otimes \tau^{(2^k-2)-2a}) = \rho_{2^d} \tau^{2a} \otimes \text{Sq}^{2^p} \tau^{2k-2-2a} = 0.
\]

Here \( \tau^{2a} \otimes \tau^{2^k+2^p-2-2a} \neq 0 \) since by (3) and (4)
\[
2b > 2^d - 2a \\
\geq 2 \cdot 2^k - 2a \\
> 2^k + 2^p - 2 - 2a.
\]

Thus \( \rho_{2^d} = 0 \), that is, \( x^*(\alpha_{2^k-2}) = 0 \) as asserted.

Next we shall prove b). Let \( x^*(\alpha_{2^k-1-2}) = \tau^{2r-2} \otimes \tau^{2k-1-2r}, r + 2 \leq k \leq d - 1 \) and \( 2a = 2^r - 2 \). Then
\[
\text{Sq}^i x^*(\alpha_{2^k-1-2}) = \tau^{2r-2} \otimes \text{Sq}^i (\tau^{2k-1-2r}) = (\tau^{2k-1-2r})^{2r-2} \otimes \tau^{2k-1-2r+i}.
\]

Here we remark that \( r \geq 2 \). For, if \( r = 2 \), by a) \( x^*(\alpha_{2^k-2}) = 0 \) for \( 3 \leq i \leq d - 1 \). Thus \( \text{Sq}^i x^*(\alpha_{2^k-1-2}) = 0 \) and we obtain
\[
\text{Sq}^1 x^*(\alpha_{2^k-2}) = x^*(\text{Sq}^1 \alpha_{2^k-2}) = 0, \\
\text{Sq}^2 x^*(\alpha_{2^k-2}) = x^*(\alpha^{2^{k-1}}) = 0, \\
\text{Sq}^4 x^*(\alpha_{2^k-2}) = \text{Sq}^{2^{k-1}} \text{Sq}^4 x^*(\alpha_{2^k-1-2}) = 0.
\]
Then it follows from the previous argument in a) that

\[ x^*(\alpha_{2^k - 2}) = \rho \tau^{2^{r-2}} \otimes \tau^{2^k - 2^r}, \]

where \( \rho \in \mathbb{Z}/2\mathbb{Z} \).

Next using \( \text{Sq}^{2^r} \), we have

\[
\text{Sq}^{2^r} x^*(\alpha_{2^k - 2}) = \rho \text{Sq}^{2^r} (\tau^{2^{r-2}} \otimes \tau^{2^k - 2^r}) = \rho \tau^{2^{r-2}} \otimes \tau^{2^k},
\]

while

\[
\text{Sq}^{2^r} x^*(\alpha_{2^k - 2}) = x^*(\alpha_{2^k + 2^{r-2}})
\]

\[
= x^*(\text{Sq}^{2^{k-1}} \text{Sq}^r \alpha_{2^{k-1} - 2})
\]

\[
= \text{Sq}^{2^{k-1}} \text{Sq}^{2^r} x^*(\alpha_{2^{k-1} - 2})
\]

\[
= \tau^{2^{r-2}} \otimes \tau^{2^k}.
\]

Here \( \tau^{2^{r-2}} \otimes \tau^{2^k} \neq 0 \) since

\[
2a = 2^r - 2 \\
2b = 2(a + b) - 2a \\
> 2^d - 2^r + 2 \\
\geq 2^{d-1} \\
\geq 2^k.
\]

Therefore \( \rho = 1 \) and

\[ x^*(\alpha_{2^k - 2}) = \tau^{2^{r-2}} \otimes \tau^{2^k - 2^r}. \]

Thus lemma 4.11 is proved.

In the case I) if \( x^*(\alpha_6) = 0 \) then \( x^*(\alpha_{2(a+b)}) = 0 \). Proof. By Lemma 4.11

\[ x^*(\alpha_{2^{d-1} - 2}) = 0 \]

or

\[ 2a = 2^r - 2 \text{ and } x^*(\alpha_{2^{d-1} - 2}) = \tau^{2^{r-2}} \otimes \tau^{2^{d-1} - 2^r}. \]

Since

\[ x^*(\alpha_{2(a+b)}) = \text{Sq}^{2^{d-1}} \text{Sq}^r \alpha_{2^{d-1} - (2^{d-2})} x^*(\alpha_{2^{d-1} - 2}), \]
if \( x^*(\alpha_{2(a+b)}) \neq 0 \) then \( x^*(\alpha_{2^r-1-2}) \neq 0 \) and \( 2(a + b) \equiv -2 \mod 2^r \). But if \( 2(a + b) \equiv -2 \mod 2^r \) then
\[
\binom{2(a+b)}{2a} \equiv \binom{2(a+b)}{2^r-2} \equiv 1 \mod 2.
\]
Thus if \( \binom{2(a+b)}{2a} \equiv 0 \mod 2 \) and \( x^*(\alpha_{2}) = 0 \) then
\[
x^*(\alpha_{2(a+b)}) = 0.
\]
Q.E.D.

Now we consider the case II) we start from the next lemma. Assume \( i + j = 2^d - 2 \) for some \( d \in \mathbb{N}, d > 3 \), \( i \) and \( j \) are even, \( i, j \geq 2 \) and
\[
i = \sum_{k=1}^{d-1} \epsilon_k 2^k,
\]
where \( \epsilon_k = 0 \) or 1. Then
\[
\text{Sq}^{2p} \tau^i \otimes \tau^j = \begin{cases} \tau^{i+2p} \otimes \tau^j & \epsilon_p = 1 \\ \tau^i \otimes \tau^{j+2p} & \epsilon_p = 0 \end{cases}
\]
for \( 1 \leq p \leq d - 1 \) where \( \tau^i \otimes \tau^j \in H^{2^d-2}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty) \). Proof. We use induction. Let \( \tau^i = 1 - \epsilon_k \). Then \( j = \sum_{k=1}^{d-1} \epsilon_k 2^k \).

The statement is true for \( p = 1 \). Let we assume that the statement is true for \( \text{Sq}^{2p-1} \) and also \( \epsilon_{p-1} = 1 \). Then
\[
\text{Sq}^{2p-1} \tau^i \otimes \tau^j = \sum_{l=0}^{2p-1} (\text{Sq}^l \tau^i \otimes \text{Sq}^{2p-1-l} \tau^j)
\]
\[
= \sum_{l=0}^{2p-1} \binom{i}{l} \binom{j}{2p-1-l} \tau^{i+l} \otimes \tau^{j+2p-1-l}
\]
\[
= \tau^{i+2p-1} \otimes \tau^j,
\]
that is,
\[
\binom{i}{l} \binom{j}{2p-1-l} = \begin{cases} 0 & 0 \leq l \leq 2p-1 - 1 \\ 1 & l = 2p-1. \end{cases}
\]
Hence

\[
\text{Sq}^{2^p} \tau^i \otimes \tau^j = \sum_{l=0}^{2^p-1} \binom{i}{2^p-l} \tau^{i+l} \otimes \tau^{j+2^p-l}
\]

\[
= \sum_{l=0}^{2^p-1} \binom{j}{2^p-l} \tau^{i+2^p-l} + \sum_{l=0}^{2^p-1} \binom{i}{2^p-l} \tau^{i+2^p-l+1} \otimes \tau^{j+2^p-l-1}
\]

\[
= \sum_{l=1}^{2^p-1} \binom{i}{2^p-1} \binom{j}{2^p-1} \tau^{i+l} \otimes \tau^{j+2^p-l} + \sum_{l=0}^{2^p-1} \binom{j}{2^p-l} \tau^{i+2p-l+1} \otimes \tau^{j+2^p-l-1}
\]

[as asserted. And even if \( \epsilon_{p-1} = 1 \), it can be proved in the same manner.]

Q.E.D.

Let \( b \geq a \). In the case II), if \( x^*(\alpha_6) = 0 \), then

\[
x^*(\alpha_{2d-2}) = \begin{cases} 
0 & \text{or} \\
\rho \sum_{i=1}^{(2^d-4)/2} \tau^{2i} \otimes \tau^{(2^d-2)-2i} + \rho' \tau^{2^d-1} \otimes \tau^{2^d-1} \\
\tau^{2r-2} \otimes \tau^{2^d-2r} \text{ and } 2a = 2^r - 2, 3 \leq r \leq d - 1.
\end{cases}
\]

Proof. We start from the computation of \( x^*(\alpha_{2d-1-2}) \). By lemma 4.11

\[
x^*(\alpha_{2d-1-2}) = \begin{cases} 
0 & \text{or} \\
\tau^{2r-2} \otimes \tau^{2d-2r} \text{ in this case } 2a = 2^r - 2, 3 \leq r \leq d - 1.
\end{cases}
\]

Next we consider \( x^*(\alpha_{2d-2}) \). Since

\[
\text{Sq}^1(x^*(\alpha_{2d-2})) = x^*(\text{Sq}^1\alpha_{2d-2}) = 0, \quad (18)
\]

\[
\text{Sq}^2(x^*(\alpha_{2d-2})) = x^*(\alpha_2^{2d-1}) = 0, \quad (19)
\]

\[
\text{Sq}^4(x^*(\alpha_{2d-2})) = \text{Sq}^{2d-1}\text{Sq}^4x^*(\alpha_{2d-1-2}) = 0, \quad (20)
\]

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as in the proof of Lemma 4.11, we have

\[ x^*(\alpha_{2d-2}) = \rho \tau^{2s} \otimes \tau^{(2d-2)-2s} + \rho' \sum_{i=s+1}^{t-1} \tau^{2i} \otimes \tau^{(2d-2)-2i} + \rho'' \tau^{2t} \otimes \tau^{(2d-2)-2t}, \]

where

\[ s = \max\{1, \frac{2d-2}{2} - b\}, \]
\[ t = \min\{\frac{2d-4}{2}, a\}. \]

Firstly we assume \( x^*(\alpha_{2d-1-2}) = 0 \). And we shall prove \( \rho = \rho' \). If \( s = 1 \) then the equation \( \text{Sq}^2x^*(\alpha_{2d-2}) = 0 \) means \( \rho = \rho' \). Thus we assume \( s = \frac{2d-2}{2} - b \), that is,

\[ 2b \leq 2^d - 4. \]  \hfill (21)

Here we remark that by (4),

\[ 2b \geq a + b \]  \hfill (22)
\[ > 2^{d-1} - 2 \)  \hfill (23)

Let \( q := p(2b) \) then (21) and (23) mean \( q \leq d - 2 \). Also

\[ \text{Sq}^2q^x(\alpha_{2d-2}) = \text{Sq}^{2d-1} \text{Sq}^q x^*(\alpha_{2d-1-2}) = 0. \]

Thus, by Lemma 4.13, compare the term of \( \tau^{(2d-2)-2b+2q} \otimes \tau^{2b} \) in \( \text{Sq}^2q^x(\alpha_{2d-2}) \) and we obtain

\[ (\rho + \rho') \tau^{(2d-2)-2b+2q} \otimes \tau^{2b} = 0. \]  \hfill (24)

Here we remark that \( (2d-2) - 2b + 2q \leq 2a \) by (4). Thus (24) means \( \rho' = \rho'' \). Therefore

\[ x^*(\alpha_{2d-2}) = \rho' \sum_{i=s}^{t-1} \tau^{2i} \otimes \tau^{(2d-2)-2i} + \rho'' \tau^{2t} \otimes \tau^{(2d-2)-2t}. \]

Next we consider the term \( \rho'' \tau^{2t} \otimes \tau^{(2d-2)-2t} \). If \( 2t = 2^d - 4 \), then by the computation of \( \text{Sq}^2x^*(\alpha_{2d-2}) \) we have \( \rho' = \rho'' \) and \( x^*(\alpha_{2d-2}) = \sum_{i=s}^{t} \tau^{2i} \otimes \tau^{(2d-2)-2i} \) or 0 as asserted. Thus we assume \( 2t = 2a \), that is,

\[ 2a < 2^d - 4 \]  \hfill (25)
Let \( p := p(2a) \). Here from (25) \( p \leq d - 1 \). And \( p = d - 1 \) if and only if \( 2a = 2^{d-1} - 2 \).

If \( 2a = 2^{d-1} - 2 \) then
\[
x^*(\alpha_{2d-2}) = \rho' \sum_{i=1}^{(2^{d-1}/2 - 2)} \tau^{2i} \otimes \tau^{(2^{d-2})-2i} + (\rho'' + \rho')\tau^{2^{d-1} - 2} \otimes \tau^{2^{d-1}}.
\]

If \( p \leq d - 2 \) then
\[
\text{Sq}^p x^*(\alpha_{2d-2}) = \text{Sq}^{2d-1} \text{Sq}^p x^*(\alpha_{2d-1-2}) = 0.
\] (26)

By Lemma 4.13 look into the term of \( \tau^{2a} \otimes \tau^{(2^{d-2})-2a+2p} \) of (26) and we obtain
\[
(\rho' + \rho'')\tau^{2a} \otimes \tau^{(2^{d-2})-2a+2p} = 0.
\] (27)

Remark that by (4)
\[
(2^d - 2) - 2a + 2^p \leq 2b.
\]

Therefore \( \rho' = \rho'' \) and
\[
x^*(\alpha_{2d-2}) = \rho' \sum_{i=s}^{t} \tau^{2i} \otimes \tau^{(2^{d-2})-2i}.
\]

Secondly we assume \( x^*(\alpha_{2d-1-2}) = \tau^{2^{d-2}} \otimes \tau^{2^{d-1} - 2} \) and \( 2a = 2^r - 2 \) and observe \( x^*(\alpha_{2d-2}) \) again. We reset
\[
x^*(\alpha_{2d-2}) = \rho \tau^{2s} \otimes \tau^{(2^{d-2})-2s} + \rho' \sum_{i=s+1}^{t-1} \tau^{2i} \otimes \tau^{(2^{d-2})-2i} + \rho'' \tau^{2t} \otimes \tau^{(2^{d-2})-2t},
\]
where
\[
s = \max\{1, \frac{2^d - 2}{2} - b\},
\]
\[
t = \min\{\frac{2^d - 4}{2}, a\}.
\]

Then
\[
2b = 2(a + b) - 2a \geq (3 \cdot 2^{d-1} - 2) - (2^{d-1} - 2) = 2^d.
\] (28) (29) (30)
This means \( s = 1 \). Thus by the computation of \( \text{Sq}^2 x^*(\alpha_{2d-2}) \) we have
\[
\rho = \rho'
\]
and also by the computation of \( \text{Sq}^4 x^*(\alpha_{2d-2}) \) and by (30) we have
\[
\rho = 0.
\]
Therefore we obtain
\[
x^*(\alpha_{2d-2}) = \rho'' \tau^{2r-2} \otimes \tau^{2d-2r}.
\]
Finally we have obtained the following result
\[
x^*(\alpha_{2d-2}) = \left\{
\begin{array}{ll}
\rho \sum_{i=1}^{(2d-4)/2} \tau^{2i} \otimes \tau^{(2d-2)-2i} + \rho' \tau^{2d-1-2} \otimes \tau^{2d-1} & \text{where } 2a = 2^{d-1} - 2 \text{ if } \rho' = 1 \\
n & \text{or} \\
\tau^{2r-2} \otimes \tau^{2d-2r} & \text{and } 2a = 2^r - 2, \ 3 \leq r \leq d - 2.
\end{array}
\right.
\]
Q.E.D.

In the case II) if \( x^*(\alpha_b) = 0 \) then \( x^*(\alpha_{2(a+b)}) = 0 \).

Proof. By (4)
\[
x^*(\alpha_{2(a+b)}) = \text{Sq}^{2(a+b)-(2d-2)} x^*(\alpha_{2d-2}).
\]
And by Lemma 4.14 we shall prove that
\[
\left\{
\begin{array}{l}
\text{Sq}^{2(a+b)-(2d-2)} \left( \sum_{i=1}^{(2d-4)/2} \tau^{2i} \otimes \tau^{(2d-2)-2i} \right) = 0 \\
\text{Sq}^{2(a+b)-(2d-2)} (\tau^{2r-2} \otimes \tau^{2d-2r}) = 0 \text{ in case } a = 2^r - 2, \ 3 \leq r \leq d - 1.
\end{array}
\right.
\]
Since
\[
\sum_{i=1}^{(2d-4)/2} \tau^{2i} \otimes \tau^{(2d-2)-2i} = x^*_0(\alpha_{2d-2}),
\]
it follows that
\[
\text{Sq}^{2(a+b)-(2d-2)} \left( \sum_{i=1}^{(2d-4)/2} \tau^{2i} \otimes \tau^{(2d-2)-2i} \right) = \text{Sq}^{2(a+b)-(2d-2)} x^*_0(\alpha_{2d-2})
\]
\[
= x^*_0(\alpha_{2(a+b)})
\]
\[
= \left( \frac{2(a+b)}{2a} \right) \tau^{2a} \otimes \tau^{2b}
\]
\[
= 0.
\]
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Also
\[
\begin{align*}
\text{Sq}^{2(a+b)-(2^d-2)}(\tau^{2r-2} \otimes \tau^{2^d-2^r}) & = \tau^{2r-2} \otimes \left(\frac{2^d-2^r}{2(a+b)-(2^d-2)}\right)^{2(a+b)-(2^d-2)} \\
& = \begin{cases} 
\tau^{2r-2} \otimes \tau^{2(a+b)-(2^d-2)} & \text{if } 2(a+b) \equiv -2 \mod 2r \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

But if \(2(a+b) \equiv -2 \mod 2r\) then
\[
\begin{align*}
\left(\frac{2(a+b)}{2^d}\right)^{2(a+b)-(2r-2)} & \equiv 1 \mod 2.
\end{align*}
\]
Thus if \(\left(\frac{2(a+b)}{2^d}\right) \equiv 0 \mod 2\) then \(x^*(\alpha_{2(a+b)}) = 0\).

Q.E.D.

Now we shall finish the proof of Theorem 4.7. Let \(x : \mathbb{RP}^{n-1} \wedge \mathbb{RP}^{m-1} \to \Omega_0\text{SO}\) be an arbitrary map, \(n > 1, m > 1\) and \(\left(\frac{n+m-2}{n-1}\right) \equiv 0 \mod 2\). If \(x^*(\alpha_{6}) = 0\) then by Lemma 4.12, Lemma 4.15 we obtain \(x^*(\alpha_{n+m-2}) = 0\). Therefore we assume \(x^*(\alpha_{6}) \neq 0\). Then from Lemma 4.9
\[
x^*(\alpha_{6}) = \tau^2 \otimes \tau^4 + \tau^4 \otimes \tau^2.
\]
Let \(x + x_0 : \mathbb{RP}^{n-1} \wedge \mathbb{RP}^{m-1} \to \Omega_0\text{SO}\) be a map which is contained in the homotopy class \([x] + [x_0]\). Since \(\Omega_0\text{SO}\) is an H-space and it is known that \(\alpha_{2i} \in H^*(\Omega_0\text{SO})\) are primitive elements,
\[
(x + x_0)^*(\alpha_6) = 2(\tau^2 \otimes \tau^4 + \tau^4 \otimes \tau^2) = 0.
\]
Therefore
\[
(x + x_0)^*(\alpha_{n+m-2}) = 0,
\]
while
\[
\begin{align*}
(x + x_0)^*(\alpha_{n+m-2}) & = x^*(\alpha_{n+m-2}) + x_0^*(\alpha_{n+m-2}) \\
& = x^*(\alpha_{n+m-2}) + \left(\frac{n+m-2}{n-1}\right)^{n-1} \otimes \tau^{m-1} \\
& = x^*(\alpha_{n+m-2}).
\end{align*}
\]
Finally we obtained that \(x^*(\alpha_{n+m-2}) = 0\) and Theorem 4.7 is proved.

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References


