ADJOINT ACTION ON HOMOLOGY MOD 2 OF E_8 ON ITS LOOP SPACE

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1. Introduction

Assume G is a compact, connected, simply connected Lie group. The space of free loops on G is called $LG\,(G)$ the free loop group of G, whose multiplication is defined as

$$\varphi \cdot \psi(t) = \varphi(t) \cdot \psi(t).$$

Let ΩG be the space of based loops on G, whose base point is the unit e. Then $LG\,(G)$ has ΩG as its normal subgroup and

$$LG\,(G)\,/\Omega G \cong G.$$

Identifying elements of G with constant maps from S^1 to G, $LG\,(G)$ is equal to the semidirect product of G and ΩG . Thus the mod p homology of $LG\,(G)$ is determined by the mod p homology of G and ΩG and the algebra structure of $\mathrm{H}_*(LG\,(G)\,; \mathbf{Z}/p\mathbf{Z})$ depends on $\mathrm{H}_*(\mathrm{ad}; \mathbf{Z}/p\mathbf{Z})$ where

$$\mathrm{ad} : G \times \Omega G \to \Omega G$$

is the adjoint map.

In [4] some properties of ad_* are studied and it is showed that $\mathrm{H}_*(\mathrm{ad}; \mathbf{Z}/p\mathbf{Z})$ is equal to $\mathrm{H}_*(p_2; \mathbf{Z}/p\mathbf{Z})$ where p_2 is the second projection if and only if $\mathrm{H}^*(G; \mathbf{Z})$ is p-torsin free. For an exceptional Lie group G, $\mathrm{H}^*(G; \mathbf{Z})$ has p-torsion when

$$\begin{array}{lcll} G &=& G_2, F_4, E_6, E_7, E_8 & \text{for } p = 2, \\ G &=& F_4, E_6, E_7, E_8 & \text{for } p = 3, \\ G &=& E_8 & \text{for } p = 5. \end{array}$$

The case where $p = 2$ and $G \neq E_8$ is discussed in [6] and the case of $p = 3, 5$ is studied in [8, 7] respectively. In this paper we offer the result of the remained case, $(G, p) = (E_8, 2)$. The result is showed in Theorem 4.1.

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This paper is organized as follows. In §2 we refer to the result of the algebra structure of $\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$ and the Hopf algebra structure and cohomology operations of them . And in §3 we introduce the adjoint action and observe its property. Finally in §4 the induced homomorphism from the adjoint action of E_8 is determined by using the result of E_7 and cohomology operations.

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2. $\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$

We refer to the result of [1] and [2] about $\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z})$ for $G = E_7$ and E_8.

Theorem 2.1.

$$\begin{aligned}\mathrm{H}^*(E_7;\mathbf{Z}/2\mathbf{Z}) &= \mathbf{Z}/2\mathbf{Z}[x_3,x_5,x_9]/(x_3^4,x_5^4,x_9^4)\otimes\bigwedge(x_{15},x_{17},x_{23},x_{27})\\ \mathrm{H}^*(E_8;\mathbf{Z}/2\mathbf{Z}) &= \mathbf{Z}/2\mathbf{Z}[x_3,x_5,x_9,x_{15}]/({x_3}^{16},{x_5}^{8},{x_9}^{4},{x_{15}}^{4})\otimes\bigwedge(x_{17},x_{23},x_{27},x_{29})\end{aligned}$$

where x_i is a generator of degree i. Moreover there is a homomorphism

$$E_7\to E_8$$

whose induced homomorphism maps x_i in $\mathrm{H}^*(E_8;\mathbf{Z}/2\mathbf{Z})$ into x_i in $\mathrm{H}^*(E_7;\mathbf{Z}/2\mathbf{Z})$.

Theorem 2.2. *The x_i's in Theorem 2.1 can be chosen so as to satisfy*

$$\begin{gathered}x_5=\mathrm{Sq}^2x_3,\\ x_9=\mathrm{Sq}^4x_5,\\ \overline{\psi}(x_3)=\overline{\psi}(x_5)=\overline{\psi}(x_9)=0\end{gathered}$$

and the coproduct of x_{15} is

$$\overline{\psi}(x_{15})={x_3}^2\otimes x_9+{x_5}^2\otimes x_5+{x_3}^4\otimes x_3.$$

The algebra structure of $\mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$ can be determined as an application of the Eilenberg-Moore spectral sequence. And the Hopf algebra structures and the action of cohomology operations which acts on homology dually was determined by A.Kono and K.Kozima. See [5, 3] for detail.

Theorem 2.3.

$$\begin{aligned}\mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z}) &= \bigwedge(b_2,b_4,b_8)\otimes\mathbf{Z}/2\mathbf{Z}[b_{10},b_{14},b_{16},b_{18},b_{22},b_{26},b_{34}]\\ \mathrm{H}_*(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z}) &= \bigwedge(b_2,b_4,b_8,b_{14})\otimes\mathbf{Z}/2\mathbf{Z}[b_{16},b_{22},b_{26},b_{28},b_{34},b_{38},b_{46},b_{58}]\end{aligned}$$

where b_i is a generator of degree i.

Theorem 2.4. *The coproduct of* $\mathrm{H}_*(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})$ *is given as*

$$\begin{aligned}
\overline{\phi}(b_i) &= 0 \text{ for } i=2,14,22,26,34,38,46,58,\\
\overline{\phi}(b_4) &= b_2\otimes b_2,\\
\overline{\phi}(b_8) &= b_2\otimes b_2b_4+b_4\otimes b_4+b_2b_4\otimes b_2,\\
\overline{\phi}(b_{16}) &= b_2\otimes b_2b_4b_8+b_4\otimes b_4b_8+b_2b_4\otimes b_2b_8+b_8\otimes b_8\\
&\quad +b_2b_8\otimes b_2b_4+b_4b_8\otimes b_4+b_2b_4b_8\otimes b_2,\\
\overline{\phi}(b_{28}) &= b_{14}\otimes b_{14}.
\end{aligned}$$

3. Adjoint action

Let $\mathrm{Ad}: G\times G\to G$ and $\mathrm{ad}: G\times \Omega G\to \Omega G$ be the adjoint action of a Lie group G defined by $\mathrm{Ad}(g,h)=ghg^{-1}$ and $\mathrm{ad}(g,l)(t)=gl(t)g^{-1}$ where $g,h\in G$, $l\in\Omega G$ and $t\in[0,1]$. These induce the homomorphisms

$$\mathrm{Ad}_*:\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})\otimes \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})\to \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$$

and

$$\mathrm{ad}_*:\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})\otimes \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})\to \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}).$$

Put $y*y'=\mathrm{Ad}_*(y\otimes y')$ and $y*b=\mathrm{ad}_*(y\otimes b)$ where $y,y'\in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ and $b\in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$. Following are the dual statement of the result in [4].

Theorem 3.1. *For* y, y', $y''\in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ *and* b, $b'\in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$

(i) $1*y=y$, $1*b=b$.

(ii) $y*1=0$, *if* $|y|>0$, *whether* $1\in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ *or* $1\in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$.

(iii) $(yy')*b=y*(y'*b)$.

(iv) $y*(bb')=\sum(y'*b)(y''*b')$ *where* $\Delta_*y=\sum y'\otimes y''$.

(v) $\sigma(y*b)=y*\sigma(b)$ *where* σ *is the homology suspension.*

(vi) $\mathrm{Sq}^n_*(y*b)=\sum_i(\mathrm{Sq}^i_*y)*(\mathrm{Sq}^{n-i}_*b)$.
$\mathrm{Sq}^n_*(y*y')=\sum_i(\mathrm{Sq}^i_*y)*(\mathrm{Sq}^{n-i}_*y')$.

(vii)

$$\begin{aligned}
\Delta_*(y*b) &= (\Delta_*y)*(\Delta_*b)\\
&= \sum(y'*b')\otimes(y''*b'')
\end{aligned}$$

where $\Delta_*y=\sum y'\otimes y''$ *and* $\Delta_*b=\sum b'\otimes b''$. *Also*

$$\overline{\Delta}_*(y*b)=(\Delta_*y)*(\overline{\Delta}_*b).$$

(viii) *If* b *is primitive then* $y*b$ *is primitive.*

Let $y_{2i} \in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ be the dual of ${x_i}^2$ for $i = 3, 5, 9, 15$ and y_{12}, y_{24}, y_{20} be the dual of ${x_3}^4, {x_3}^8, {x_5}^4$ respectively with respect to the monomial basis. Also in $\mathrm{H}_*(E_8;\mathbf{Z}/2\mathbf{Z})$ we put as

$$y^m = {y_6}^{m_1}{y_{12}}^{m_2}{y_{24}}^{m_3}{y_{10}}^{m_4}{y_{20}}^{m_5}{y_{18}}^{m_6}{y_{30}}^{m_7}$$

for $m = (m_1, m_2, \cdots, m_7) \in \mathbf{Z}/2\mathbf{Z}^7$. Then the result of [4] implies the next theorem. See [6].

Theorem 3.2. *We define a submodule A of $\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ as*

$$\begin{array}{rcll} A & = & \bigwedge(y_6, y_{10}, y_{18}) & \text{for } G = E_7 \\ A & = & \langle y^m \text{ for all } m \in \mathbf{Z}/2\mathbf{Z}^7 \rangle & \text{for } G = E_8. \end{array}$$

Then there exist a retraction $p : \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \to A$ and the following diagram commutes.

$$\begin{tikzcd} \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \arrow[r, "ad_*"] \arrow[d, "p \otimes 1"] & \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \\ A \otimes \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \arrow[ur, "ad_*"] & \end{tikzcd}$$

Remark

1. The submodule A has an algebra structure induced from that of $\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$. When $G = E_7$, A is a commutative exterior algebra over $\mathbf{Z}/2\mathbf{Z}$. But when $G = E_8$, A is a non-commutative algebra over $\mathbf{Z}/2\mathbf{Z}$. In fact A is the dual of $\bigwedge({x_3}^2, {x_5}^2, {x_9}^2)$ for $G = E_7$ and is the dual of $\mathbf{Z}/2\mathbf{Z}[{x_3}^2, {x_5}^2, {x_9}^2, x_{15}^2]/({x_3}^{16}, {x_5}^8, {x_9}^4, {x_{15}}^4)$ for $G = E_8$. Thus we can easily see that, for $G = E_8$, A is generated by $\{y_6, y_{12}, y_{24}, y_{10}, y_{20}, y_{18}\}$ as algebra and the fundamental relations are

$$y_{2i}^2 = 0 \text{ for } i = 3, 6, 12, 5, 10, 9,$$

$$[y_{2i}, y_{2j}] = 0 \text{ for } (i, j) \neq (6, 9), (9, 6), (5, 10), (10, 5), (3, 18), (18, 3)$$

and

$$[y_6, y_{24}] = [y_{10}, y_{20}] = [y_{12}, y_{18}] (= y_{30}).$$

2. By Theorem 3.1 (iv) and Theorem 3.2 we see that for $b \in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$ and $i = 3, 5, 9$

$$\begin{array}{rcl} y_{2i} * b^2 & = & (y_{2i} * b)b + (y_i * b)^2 + b(y_{2i} * b) \\ & = & 0 \end{array}$$

where y_i is the dual of x_i for $i = 3, 5, 9$ with respect to the monomial basis.

3. By theorem 3.1 and 3.2, when $G = E_8$, if $y_i * b_j$ is determined for $i = 6, 12, 24, 10, 20, 18$ and $b_j \in \mathrm{H}_*(G; \mathbf{Z}/2\mathbf{Z})$, then the map $\mathrm{H}_*(\mathrm{ad}; \mathbf{Z}/2\mathbf{Z})$ is determined completely.

4. Adjoint action on ΩE_8

The next theorem is the main result of this paper.

Theorem 4.1. *For $j \in \{6, 12, 24, 10, 20, 18\}$ and $b_i \in \mathrm{H}_*(\Omega E_8\ ; \mathbf{Z}/2\mathbf{Z})$, $y_j * b_i$ is given by the following tables.*

b_j	$y_6 * b_j$	$y_{10} * b_j$	$y_{18} * b_j$
b_2	0	0	0
b_4	0	b_{14}	b_{22}
b_8	b_{14}	$b_4 b_{14}$	$b_{26} + b_4 b_{22}$
b_{14}	0	0	b_{16}^2
b_{16}	$b_{22} + b_8 b_{14}$	$b_{26} + b_4 b_8 b_{14}$	$b_{34} + b_8 b_{26} + b_4 b_8 b_{22}$
b_{22}	b_{14}^2	b_{16}^2	0
b_{26}	b_{16}^2	0	b_{22}^2
b_{28}	b_{34}	b_{38}	$b_{16}^2 b_{14} + b_{46}$
b_{34}	0	b_{22}^2	b_{26}^2
b_{38}	b_{22}^2	0	b_{28}^2
b_{46}	b_{26}^2	b_{28}^2	b_{16}^4
b_{58}	b_{16}^4	b_{34}^2	b_{38}^2

b_j	$y_{12} * b_j$	$y_{20} * b_j$	$y_{24} * b_j$
b_2	b_{14}	b_{22}	b_{26}
b_4	$b_2 b_{14}$	$b_2 b_{22}$	$b_{28} + b_2 b_{26}$
b_8	$b_2 b_4 b_{14}$	$b_{28} + b_2 b_4 b_{22}$	$b_4 b_{28} + b_2 b_4 b_{26}$
b_{14}	0	b_{34}	b_{38}
b_{16}	$b_{28} + b_2 b_4 b_8 b_{14}$	$b_8 b_{28} + b_2 b_4 b_8 b_{22}$	$b_4 b_8 b_{28} + b_2 b_4 b_8 b_{26}$
b_{22}	b_{34}	0	b_{46}
b_{26}	b_{38}	b_{46}	0
b_{28}	0	0	b_{26}^2
b_{34}	0	0	b_{58}
b_{38}	0	b_{58}	0
b_{46}	b_{58}	0	0
b_{58}	0	0	0

Remark The action of cohomology operations on $\mathrm{H}_*(\Omega E_8\ ; \mathbf{Z}/2\mathbf{Z})$ is determined by A.Kono and K.Kozima in [3]. But we do not use them. We use the Hopf algebra structure of $\mathrm{H}_*(\Omega E_8\ ; \mathbf{Z}/2\mathbf{Z})$ and the result in $\mathrm{H}_*(\Omega E_7\ ; \mathbf{Z}/2\mathbf{Z})$.

Proof. In $\mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z})$ $y_j * b_i$ and $\mathrm{Sq}_*^{2^k} b_i$ are determined as follows. See Theorem 5.11 in [6].

b_i	$y_6 * b_i$	$y_{10} * b_i$	$y_{18} * b_i$
b_2	0	0	b_{10}^2
b_4	b_{10}	b_{14}	$b_{22} + b_2 b_{10}^2$
b_8	$b_{14} + b_4 b_{10}$	$b_{18} + b_4 b_{14}$	$b_{26} + b_4 b_{22} + b_2 b_4 b_{10}^2$
b_{10}	0	b_{10}^2	b_{14}^2
b_{14}	b_{10}^2	0	b_{16}^2
b_{16}	$b_{22} + b_8 b_{14} + b_4 b_8 b_{10}$	$b_{26} + b_8 b_{18} + b_4 b_8 b_{14}$	$b_{34} + b_8 b_{26} + b_4 b_8 b_{22} + b_2 b_4 b_8 b_{10}^2$
b_{18}	0	b_{14}^2	b_{18}^2
b_{22}	b_{14}^2	b_{16}^2	b_{10}^4
b_{26}	b_{16}^2	b_{18}^2	b_{22}^2
b_{34}	b_{10}^4	b_{22}^2	b_{26}^2

b_i	$\mathrm{Sq}_*^2 b_i$	$\mathrm{Sq}_*^4 b_i$	$\mathrm{Sq}_*^8 b_i$	$\mathrm{Sq}_*^{16} b_i$
b_4	b_2	—	—	—
b_8	$b_2 b_4$	b_4	—	—
b_{10}	b_4^2	0	—	—
b_{14}	0	b_{10}	—	—
b_{16}	$b_{14} + b_2 b_4 b_8$	$b_4 b_8$	b_8	—
b_{18}	0	0	b_{10}	—
b_{22}	b_{10}^2	0	b_{14}	—
b_{26}	0	b_{22}	b_{18}	—
b_{34}	b_{16}^2	0	0	b_{18}

By the naturality of adjoint action, the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{H}_*(E_7;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z}) & \xrightarrow{\mathrm{ad}_*} & \mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z}) \\
\downarrow & & \downarrow \\
\mathrm{H}_*(E_8;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z}) & \xrightarrow{\mathrm{ad}_*} & \mathrm{H}_*(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})
\end{array}$$

Thus we can easily see that above tables remain true also in $\mathrm{H}_*(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})$ except for $y_j * b_{10}$ and $y_j * b_{18}$ by replacing b_{10}, b_{18} by 0.

Also we can easily see that

$$\mathrm{Sq}_*^8 \mathrm{Sq}_*^4 \mathrm{Sq}_*^2 b_{28} = \mathrm{Sq}_*^{14} b_{28} = b_{14} \neq 0.$$

This means $\mathrm{Sq}_*^2 b_{28} = b_{26}$.

If b_i is primitive, $y_j * b_i$ is primitive. By (viii) of Theorem 3.1, $y_j * b_i$ is primitive for

$$(i,j) \in \left\{ \begin{array}{l} (10,38),(12,38),(12,58),(20,22),(20,34),(20,46), \\ (20,58),(24,26),(24,38),(24,46),(24,58) \end{array} \right\}.$$

Since no primitive elements of these degrees are there in $\mathrm{H}_*(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})$, these elements are 0.

Next we consider $y_{12} * b_2$. Because $y_{12} * b_2$ is primitive, it is b_{14} or 0. On the other hand, we have

$$\begin{aligned}\overline{\Delta}_*(y_{12} * b_4) &= (y_{12} * b_2) \otimes b_2 + (y_6 * b_2) \otimes (y_6 * b_2) + b_2 \otimes (y_{12} * b_2) \\ &= \overline{\Delta}_*((y_{12} * b_2)b_2).\end{aligned}$$

This means $y_{12} * b_4 = (y_{12} * b_2)b_2$ since there is no primitive element in $\mathrm{H}_{16}(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})$. Therefore we have

$$\mathrm{Sq}^2_*(y_{12} * b_4) = \mathrm{Sq}^2_*(y_{12} * b_2)b_2 = 0,$$

while

$$\mathrm{Sq}^2_*(y_{12} * b_4) = y_{10} * b_4 + y_{12} * b_2 = b_{14} + y_{12} * b_2.$$

Hence we obtain

$$\begin{aligned}y_{12} * b_2 &= b_{14}, \\ y_{12} * b_4 &= b_{14}b_2.\end{aligned}$$

In the same way we can easily show

$$\begin{aligned}y_{20} * b_2 &= b_{22}, \\ y_{20} * b_4 &= b_{22}b_2, \\ y_{24} * b_2 &= b_{26}, \\ y_{24} * b_4 &= b_{28} + b_{26}b_2.\end{aligned}$$

Since

$$\overline{\Delta}_*(y_{12} * b_8) = \Delta_*(y_{12}) * \overline{\Delta}_* b_8 = \overline{\Delta}_*(b_{14}b_4b_2)$$

and no primitive element is there in $\mathrm{H}_{20}(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})$, we have

$$y_{12} * b_8 = b_{14}b_4b_2.$$

In the similar way we can determine

$$y_{12} * b_{28},\ y_{20} * b_{28},\ y_{12} * b_{16},\ y_{20} * b_8,\ y_{20} * b_{16}$$

as in the table of Theorem.

Also as

$$\begin{aligned}\overline{\Delta}_*(y_{24} * b_8) &= \Delta_* y_{24} * \overline{\Delta}_* b_8 \\ &= \overline{\Delta}_*(b_{26}b_4b_2 + b_{28}b_4)\end{aligned}$$

and the only primitive element in $\mathrm{H}_{32}(\Omega E_8\ ;\mathbf{Z}/2\mathbf{Z})$ is ${b_{16}}^2$, we can put

$$y_{24} * b_8 = b_{26}b_4b_2 + b_4b_{28} + \rho {b_{16}}^2 \tag{1}$$

where $\rho \in \mathbf{Z}/2\mathbf{Z}$. Applying Sq^4_* to each side of (1), we have

$$\mathrm{Sq}^4_*(y_{24} * b_8) = y_{20} * b_8 + y_{24} * b_4 = b_{22}b_4b_2 + b_{26}b_2,$$

while

$$\mathrm{Sq}^4_*(b_{26}b_4b_2 + b_{28}b_4 + \rho {b_{16}}^2) = b_{22}b_4b_2 + b_{26}b_2 + \rho {b_{14}}^2.$$

Thus $\rho = 0$ and $y_{24} * b_8$ is determined. Now we can determine $y_{24} * b_{16}$ modulo primitive elements. Since no primitive elements is there in $\mathrm{H}_{40}(\Omega E_8\,;\mathbf{Z}/2\mathbf{Z})$, we can determine $y_{24} * b_{16}$ as

$$y_{24} * b_{16} = b_{28}b_8b_4 + b_{26}b_8b_4b_2.$$

Since b_{14} is primitive, $y_{20} * b_{14} = b_{34}$ or 0. Also $\mathrm{Sq}^2_*(y_{20} * b_{14}) = y_{18} * b_{14} = {b_{16}}^2$. This implies

$$y_{20} * b_{14} = b_{34},\ \mathrm{Sq}^2_* b_{34} = {b_{16}}^2.$$

In the similar way we apply Sq^2_* to $y_6 * b_{28}$, Sq^2_* to $y_{12} * b_{22}$, Sq^4_* to $y_{12} * b_{26}$ and Sq^2_* to $y_{20} * b_{26}$ and see that the followings are determined as the statement:

$$y_6 * b_{28}, y_{12} * b_{22}, y_{12} * b_{26}, y_{20} * b_{26}, \mathrm{Sq}^4_* b_{38}, \mathrm{Sq}^2_* b_{46}.$$

From the above result we can deduce that

$$\mathrm{Sq}^8_* b_{46} = \mathrm{Sq}^8_*(y_{20} * b_{26}) = y_{12} * b_{26} = b_{38}.$$

Also as $\overline{\Delta}_* \mathrm{Sq}^4_* b_{28} = \mathrm{Sq}^4_* \overline{\Delta}_* b_{28} = 0$, we have $\mathrm{Sq}^4_* b_{28} = 0$. In the similar way we have

$$\mathrm{Sq}^{2^k}_* b_i = 0 \text{ for } (k,j) \in \left\{ \begin{array}{l} (3,28), (1,38), (3,38), (2,46), \\ (4,46), (2,58), (3,58), (4,58) \end{array} \right.$$

Using the above result we can compute $\mathrm{Sq}^4_*(y_{18} * b_{38})$ as

$$\mathrm{Sq}^4_* y_{18} * b_{38} = y_{18} * b_{34} = {b_{26}}^2,$$

while $y_{18} * b_{38} = {b_{28}}^2$ or 0. This implies $y_{18} * b_{38} = {b_{28}}^2$. In the similar manner, applying Sq^4_* to $y_{10} * b_{28}$, Sq^4_* to $y_{10} * b_{38}$, Sq^8_* to $y_6 * b_{46}$, Sq^2_* to $y_{12} * b_{34}$, Sq^4_* to $y_{24} * b_{14}$ and Sq^2_* to $y_{24} * b_{22}$, the followings are determined:

$$y_{10} * b_{28}, y_6 * b_{38}, y_6 * b_{46}, y_{12} * b_{34}, y_{24} * b_{14}, y_{24} * b_{22}$$

as in the table in Theorem.

Moreover by applying Sq^4_* to $y_{10} * b_{46}$, Sq^2_* to $y_{12} * b_{46}$ and Sq^2_* to $y_{20} * b_{38}$ we have that

$$\begin{aligned} y_{10} * b_{46} &= {b_{28}}^2, \\ y_{12} * b_{46} &= b_{58}, \\ y_{20} * b_{38} &= b_{58}. \end{aligned}$$

Since ${y_{18}}^2 * b_{28} = 0$, we can see

$$y_{18} * (y_{18} * b_{28}) = y_{18} * ({b_{16}}^2 b_{14} + b_{46}) = {b_{16}}^4 + y_{18} * b_{46} = 0.$$

Therefore $y_{18} * b_{46} = {b_{16}}^4$. In this way we compute ${y_{12}}^2 * b_2$, ${y_{24}}^2 * b_4$ to obtain

$$\begin{aligned} y_{12} * b_{14} &= 0, \\ y_{24} * b_{28} &= {b_{26}}^2. \end{aligned}$$

Also we can compute $y_{24} * b_{34}$ as

$$y_{24} * b_{34} = y_{24} * (y_{20} * b_{14}) = y_{20} * (y_{24} * b_{14}) = y_{20} * b_{38} = b_{58}.$$

The rest we have to do is to determine $y_6 * b_{58}$, $y_{10} * b_{58}$ and $y_{18} * b_{58}$. By applying Sq_*^2 to $y_{20} * b_{38}$, we have

$$\mathrm{Sq}_*^2 b_{58} = \mathrm{Sq}_*^2 (y_{20} * b_{38}) = y_{18} * b_{38} = {b_{28}}^2.$$

Thus by applying Sq_*^2 to $y_{12} * b_{58}$, it follows that

$$0 = \mathrm{Sq}_*^2 (y_{12} * b_{58}) = y_{10} * b_{58} + y_{12} * {b_{28}}^2 = y_{10} * b_{58} + {b_{34}}^2.$$

Therefore $y_{10} * b_{58} = {b_{34}}^2$. We apply Sq_*^4 to $y_{10} * b_{58}$ and Sq_*^8 to $y_{18} * b_{58}$ to obtain

$$\begin{aligned} y_6 * b_{58} &= {b_{16}}^4, \\ y_{18} * b_{58} &= {b_{38}}^2. \end{aligned}$$

Now we obtain the all entries of the tables in Theorem 4.1.

Q.E.D.

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