Adjoint actions on the modulo 5 homology groups of E_8 and ΩE_8

Hiroaki HAMANAKA*
Department of Mathematics, Kyoto University,
Shin-ichiro HARA
Nagaoka University of Technology,
Akira KONO
Department of Mathematics, Kyoto University

July 10, 1997

1 Introduction

Borel proved in [2] that the integral homology group of the exceptional Lie group E_8 is not 5–torsion free and

$$H^*(E_8; \mathrm{Z}/5) \cong \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathrm{Z}/5[x_{12}]/({x_{12}}^5), \text{ with } |x_i| = i,$$

as algebra.

Araki showed the non-commutativity of the Pontrjagin ring $H_*(E_8; \mathrm{Z}/5)$ in [1]. The whole Hopf algebra structure and the cohomology operations were determined by Kono in [6]. But it was due to the partial computation of $\mathrm{Cotor}^{H^*(E_8;\mathrm{Z}/5)}(\mathrm{Z}/5, \mathrm{Z}/5)$, which was rather complicated. In [5], using secondary cohomology operations, Kane gave a general theorem to determine the Pontrjagin ring which is non-commutative and determined $H_*(E_8; \mathrm{Z}/5)$ as a Hopf algebra over $\mathcal{A}_5$.

*Partially supported by JSPS Research Fellowships for Young Scientists.

Also, for a compact, connected Lie group G, the free loop group of G denoted by $LG(G)$ is the space of free loops on G equiped with multiplication as

$$\phi \cdot \psi(t) = \phi(t) \cdot \psi(t),$$

and has ΩG as its normal subgroup. Thus

$$LG(G)/\Omega G \cong G,$$

and identifying elements of G with constant maps from S^1 to G, $LG(G)$ is equal to the semi-direct product of G and ΩG. This means that the homology of $LG(G)$ is determined by the homology of G and ΩG as module and the algebra structure of $H_*(LG(G); Z/p)$ depends on $H_*(Ad; Z/p)$ where

$$Ad : G \times \Omega G \to \Omega G$$

is the adjoint map. Since the next diagram commutes where λ,λ' and μ are the multiplication maps of ΩG, $LG(G)$ and G respectively and ω is the composition

$$(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G \times G} \times Ad \times 1_G) \circ (1_{\Omega G} \times \Delta_G \times 1_{\Omega G \times G}),$$

$$\begin{tikzcd}
\Omega G \times G \times \Omega G \times G \arrow[r, "\omega"] \arrow[d, "\cong \times \cong"] & \Omega G \times \Omega G \times G \times G \arrow[r, "\lambda \times \mu"] & \Omega G \times G \arrow[d, "\cong"] \\
LG(G) \times LG(G) \arrow[rr, "\lambda'"] & & LG(G)
\end{tikzcd}$$

we can determine directly the algebra structure of $H_*(LG(G); Z/p)$ by the knowledge of the Hopf algebra structure of $H_*(G; Z/p)$, $H_*(\Omega G; Z/p)$ and induced homology map $H_*(Ad; Z/p)$. See Theorem 6.12 of [4] for detail. Moreover, in [8], it is showed that provided G is simply connected, $H^*(Ad; Z/p)$ is equal to the induced homology map of second projection if and only if $H_*(G; Z)$ is p-torsion free. Thus the case of $(G, p) = (E_8, 5)$ is non-trivial.

In this paper we determine $H_*(Ad; Z/5)$ for $G = E_8$ and at the same time, we offer a more simple method for the determination of the coproduct and the cohomology operations on $H^*(E_8; \mathrm{Z}/5)$ using the adjoint actions of E_8 on ΩE_8. We also determine $H_*(\Omega E_8; \mathrm{Z}/5)$ as a Hopf algebra over $\mathcal{A}_5$.

This paper is organized as follows. In the next section we breifly see the algebra structures of $H^*(E_8; \mathrm{Z}/5)$ and $H_*(\Omega E_8; \mathrm{Z}/5)$ using the Serre spectral sequences. In the third section we determine the adjoint action of $H_*(E_8; \mathrm{Z}/5)$ on $H_*(\Omega E_8; \mathrm{Z}/5)$ which was introduced in [8]. It gives a easy computation of the Hopf algebra structures and the cohomology operations on them.

2 Algebra structures

Let $n(j)$, $(1 \leq j \leq 8)$, be the exponent of E_8, i.e.

$$\{n(j)\}_{1\leq j\leq 8} = \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

First we see $H^*(\Omega E_8; \mathrm{Z}/5)$ for low dimensions. Let R be the algebra $\mathrm{Z}/5[a_{2n(j)}|1 \leq j \leq 8]$ with $|a_i| = i$. By Bott ([3]), the Hopf algebra $H^*(\Omega E_8; \mathrm{Z}/5)$ is isomorphic to R as a vector space. There is a map $q : SU(9) \to E_8$ which induces an isomorphism of π_3. Then, $\Omega q : \Omega SU(9) \to \Omega E_8$ induces an isomorhpism of π_2 and, as showed in [7], $(\Omega q)^* a_2 \in H^2(\Omega SU(9); \mathrm{Z}/5)$ is nontrivial and $((\Omega q)^* a_2)^5 \neq 0$ for the generator $a_2 \in H^2(\Omega E_8; \mathrm{Z}/5)$. Thus we have ${a_2}^5 \neq 0$. It follows that $H^*(\Omega E_8; \mathrm{Z}/5)$ is isomorphic to R for $* < 50$ as algebra. Next there is two possibilities (I) : ${a_2}^{25} \neq 0$ and (II) : ${a_2}^{25} = 0$. That is, we can assume it is isomorphic to (I) : R or (II) : $R/({a_2}^{25}) \otimes \mathrm{Z}/5[a_{50}]$, for $* < 10 \cdot n(2) = 70$, where $|a_{50}| = 50$.

Consider the following Serre fibre sequences :

$$\tilde{E}_8 \xrightarrow{k} E_8 \xrightarrow{\iota} K(\mathrm{Z}, 3), \tag{1}$$

$$K(\mathrm{Z}, 1) \longrightarrow \Omega\tilde{E}_8 \xrightarrow{\Omega k} \Omega E_8, \tag{2}$$

$$\Omega\tilde{E}_8 \longrightarrow * \longrightarrow \tilde{E}_8, \tag{3}$$

where ι induces an isomorphism of π_3.

Let $\tilde{R} \equiv \mathrm{Z}/5[\tilde{a}_{2n(i)}|2 \leq i \leq 8]$ with $|\tilde{a}_i| = i$. Computing the Serre spectral sequence associated to (2), we can see that, for $* < 70$, $H^*(\Omega\tilde{E}_8; \mathrm{Z}/5)$ is isomorphic to (I) : $\tilde{R}$ or (II) : $\tilde{R} \otimes \Lambda(\tilde{a}_{49}) \otimes \mathrm{Z}/5[a_{50}]$ according to the case : ${a_2}^{25} \neq 0$ or ${a_2}^{25} = 0$. Let $\tilde{S} \equiv \Lambda(\tilde{x}_{2n(j)+1}|2 \leq j \leq 8)$ with $|\tilde{x}_i| = i$. Again computing the spectral sequence associated to (3), we have, for $* < 71$, $H^*(\tilde{E}_8; \mathrm{Z}/5)$ is isomorphic to (I) : $\tilde{S}$ or (II) : $\tilde{S} \otimes \mathrm{Z}/5[\tilde{x}_{50}] \otimes \Lambda(\tilde{x}_{51})$ where $|\tilde{x}_{50}| = 50, |\tilde{x}_{51}| = 51$.

Recall the fact :

$$H^*(K(\mathrm{Z}, 3); \mathrm{Z}/5) \cong \Lambda(u_3, u_{11}, u_{51}, \cdots) \otimes \mathrm{Z}/5[u_{12}, u_{52}, \cdots],\ |u_i| = i, \tag{4}$$

where $u_{11} = \mathcal{P}^1 u_3, u_{12} = \beta u_{11}, u_{51} = \mathcal{P}^5 u_{11}$ and $u_{52} = \beta u_{51}$.

Let $x_i = \iota^*(u_i)$, for $i = 11, 12, 51$ and 52, in $H^*(E_8; \mathrm{Z}/5)$. By the spectral sequence associated to (1), we obtain, for $* < 58$, $H^*(E_8; \mathrm{Z}/5) \cong$

(I) : $S \otimes \Lambda(x_{11}, x_{51}) \otimes \mathrm{Z}/5[x_{12}, x_{52}]$ or (II) : $S \otimes \Lambda(x_{11}) \otimes \mathrm{Z}/5[x_{12}]$, where $S \equiv \Lambda(x_{2n(j)+1} | 1 \leq j \leq 7)$ with $|x_i| = i$.

As $\dim E_8 = 248$, we can conclude that the possible case is (II) and ${x_{12}}^5 = 0$. Moreover, the generators $\{x_i\}$ are enough to generate $H^*(E_8; \mathrm{Z}/5)$. We have determined the algebra structure.

Therorem 1 *There is an algebra isomorphism :*

$$H^*(E_8; \mathrm{Z}/5) \cong \Lambda(x_{2n(j)+1} | 1 \leq j \leq 7) \otimes \Lambda(x_{11}) \otimes \mathrm{Z}/5[x_{12}]/({x_{12}}^5).$$

In $H^*(\tilde{E}_8; \mathrm{Z}/5)$, we can chose $\tilde{x}_{50}$ and $\tilde{x}_{51}$ such that $\tau' \tilde{x}_{50} = u_{51}$ and $\tau' \tilde{x}_{51} = u_{52}$, where τ' is the transgression. Then $\tau' \mathcal{P}^1 \tilde{x}_{51} = \mathcal{P}^1 u_{52} = \mathcal{P}^1 \beta \mathcal{P}^5 u_{11} = \mathcal{P}^6 \beta u_{11} = \mathcal{P}^6 u_{12} = {u_{12}}^5$. So we can chose $\tilde{x}_{59}$ as $\mathcal{P}^1 \tilde{x}_{51}$. Thus we have

Proposition 2 *There is an isomorphism for $* < 71$:*

$$H^*(\tilde{E}_8; \mathrm{Z}/5) \cong \Lambda(\tilde{x}_{2n(j)+1} | 2 \leq j \leq 8) \otimes \mathrm{Z}/5[\tilde{x}_{50}] \otimes \Lambda(\tilde{x}_{51}),$$

and

$$\mathcal{P}^1(\tilde{x}_{51}) = \tilde{x}_{59}.$$

Because that $\tilde{a}_i$ is transgressed to $\tilde{x}_{i+1}$ and $(\Omega k)^* a_i = \tilde{a}_i$ for $i = 50, 58$, the next proposition is obtained.

Proposition 3 *There are isomorphisms for $* < 70$:*

$$H^*(\Omega\tilde{E}_8; \mathrm{Z}/5) \cong \mathrm{Z}/5[\tilde{a}_{2n(j)} | 2 \leq j \leq 8] \otimes \Lambda(\tilde{a}_{49}) \otimes \mathrm{Z}/5[\tilde{a}_{50}],$$

$$H^*(\Omega E_8; \mathrm{Z}/5) \cong \mathrm{Z}/5[a_{2n(j)} | 2 \leq j \leq 8]/({a_2}^{25}) \otimes \mathrm{Z}/5[a_{50}],$$

with $\mathcal{P}^1(\tilde{a}_{50}) \equiv \tilde{a}_{58}$ *and* $\mathcal{P}^1(a_{50}) \equiv a_{58}$ *(modulo decomposable).*

By the use of a Rothenberg-Steenrod spectral sequence ([10]) :

$$E_2 \cong H^{**}(H_*(\Omega E_8; \mathrm{Z}/5)) \equiv \mathrm{Ext}_{H_*(\Omega E_8 : \mathrm{Z}/5)}(\mathrm{Z}/5, \mathrm{Z}/5) \Rightarrow E_\infty = \mathcal{G}r(H^*(E_8; \mathrm{Z}/5)),$$

it is easily seen that

Therorem 4 *There is an algebra isomorphism :*

$$H_*(\Omega E_8; \mathrm{Z}/5) \cong \mathrm{Z}/5[t_{2n(j)} | 1 \leq j \leq 8]/({t_2}^5) \otimes \mathrm{Z}/5[t_{10}].$$

(Remark) The algebra was determined first in [9].

Let σ denote the homology suspension. Examining the spectaral sequence, we have the following proposition.

Proposition 5 $\sigma(t_{2n(j)}), (1 \leq j \leq 7)$, *and* $\sigma(t_{10})$ *are nontirivial primitive elements in* $H_*(E_8; \mathrm{Z}/5)$.

3 Coproducts, cohomology operations and adjoint actions

Let $(\ )^*$ denote the dual as to the monomial basis of $\{x_i\}$ and put $y_i = (x_i)^*$.

We recall the adjoint action which was mentioned in [8]. Let $ad : G \times G \to G$ and $Ad : G \times \Omega G \to \Omega G$ be the adjoint actions for the Lie group G. Consider the induced maps of homlogy groups :

$$\begin{aligned} ad_* &: H_*(G) \otimes H_*(G) \to H_*(G), \\ Ad_* &: H_*(G) \otimes H_*(\Omega G) \to H_*(\Omega G). \end{aligned}$$

Put $y * y' = ad_*(y \otimes y')$ and $y \cdot t = yt = Ad_*(y \otimes t)$.

Our result is the following.

Therorem 6 *In $H_*(E_8; \mathrm{Z}/5)$, there are $y_{2n(j)+1}$,$(1 \leq j \leq 7)$, y_{11} and y_{12} satisfying that*

y_i	y_3	y_{11}	y_{12}	y_{15}	y_{23}	y_{27}	y_{35}	y_{39}	y_{47}
$y_{12} * y_i$	y_{15}	y_{23}	0	y_{27}	y_{35}	y_{39}	y_{47}	0	0
$\mathcal{P}^1_* y_i$	0	y_3	0	0	y_{15}	0	y_{27}	0	y_{39}
$\beta_* y_i$	0	0	y_{11}	0	0	0	0	0	0

All y_i are primitive and $y_{12} * y_i = [y_{12}, y_i] = y_{12}y_i - y_i y_{12}$.

(Remark) This result coincides with that of §46 - 2 of [5].

From now on, we prove this theorem combining the adjoint actions on $H^*(E_8; \mathrm{Z}/5)$ and $H^*(\Omega E_8; \mathrm{Z}/5)$.

Dualizing the properties of ad^* and Ad^* stated in [8], we have

Proposition 7 *For $y, y', y'' \in H_*(G)$ and $t, t', t'' \in H_*(\Omega G)$*

(1) $1 * y = y,\ 1 \cdot t = t.$

(2) $y * 1 = 0$ *and* $y \cdot 1 = 0$, *if* $|y| > 0$.

(3) $(yy')t = y(y't)$.

(4) $y(tt') = \Sigma(-1)^{|y''||t|}(y't)(y''t')$, *where* $\Delta_* y = \Sigma y' \otimes y''$ *is the coproduct.*

(5) $\phi(y \cdot t) = \Delta_*(y) \cdot \phi(t)$*, where* ϕ *is the coproduct and* $(y' \otimes y'') \cdot (t' \otimes t'') = (-1)^{|y''||t'|}(y't' \otimes y''t'')$.

(6) $\sigma(y \cdot t) = y * \sigma(t)$*, where* σ *is the homology suspension.*

(7) If y *is primitive then* $y * y' = [y, y']$*, where* $[y, y'] = yy' - (-1)^{|y||y'|}y'y$.

(8) If t *is primitive then* $y \cdot t$ *is also primitive.*

(9) $\mathcal{P}^n_*(y * y') = \Sigma_i \mathcal{P}^{n-i}_* y * \mathcal{P}^i_* y'$ *and* $\mathcal{P}^n_*(y \cdot t) = \Sigma_i \mathcal{P}^{n-i}_* y \cdot \mathcal{P}^i_* t$.

(Remark) In our case, $|t|$ and $|t'|$ are always even. So $y(tt') = \Sigma(y't)(y''t')$ and $(y' \otimes y'') \cdot (t' \otimes t'') = (y't' \otimes y''t'')$.

To state the non-commutativity of $H_*(E_8; \mathrm{Z}/5)$, we need only the fact :

Lemma 8 $[y_{12}, y_3] \neq 0$.

[Proof] Suppose that $[y_{12}, y_3] = 0$. Then $H_*(E_8; \mathrm{Z}/5) \cong \Lambda(y_3, y_{11}, y_{15}) \otimes \mathrm{Z}/5[y_{12}]$ for $* < 23$. Let $\{E'_r\}$ be the Rothenberg-Steenrod spectral sequece coversing to $H^*(BE_8; \mathrm{Z}/5)$. Then we have

$$E'_2 \cong \mathrm{Z}/5[s(y_3), s(y_{11}), s(y_{15})] \otimes \Lambda(s(y_{12}))$$

for total degree < 24. Since $E'_2 = E'_\infty$ in these degrees, there are indecomposable elements z_4, z_{12}, z_{16} and z_{13} in $H^*(BE_8; \mathrm{Z}/5)$ corresponding to $s(y_3), s(y_{11}), s(y_{15})$ and $s(y_{12})$, respectively. Especially,$z_4 z_{13} \neq 0$. It is a contradiction. (For detail, see Lemma 5.3 and 5.4 of [6].) ∎

Therefore $[y_{12}, y_3]$ is the nontrivial primitive element. So we may define y_{15} by that.

Proposition 9 $[y_{12}, y_3] = y_{15}$.

Since $\sigma(y_{12}t_2) = y_{12} * \sigma(t_2) = y_{12} * y_3 = [y_{12}, y_3] = y_{15}$, $y_{12}t_2$ is the indecomposable element. Thus we may assume that

$$t_{14} = y_{12}t_2. \tag{5}$$

Then t_{14} is primitive and $\sigma(t_{14}) = y_{15}$.

Let ϕ be the coproduct of $H_*(\Omega E_8; \mathrm{Z}/5)$ and $\overline{\phi}(t) = \phi(t) - t \otimes 1 - 1 \otimes t$. $(\ )^*$ denotes the dual as to the monomial basis of $\{t_{2j}\}$. Multiplying a_i and t_i by nonzero scalars or moving them modulo decomposable if we need, we may assume that $a_{2n(j)} = (t_{2n(j)})^*, (1 \leq j \leq 8)$, ${a_2}^5 = (t_{10})^*$ and $a_{50} = ({t_{10}}^5)^*$. As t_{10} is dual to ${a_2}^5$, it is easily verified that

$$\overline{\phi}(t_{10}) = 4{t_2}^4 \otimes t_2 + 3{t_2}^3 \otimes {t_2}^2 + 3{t_2}^2 \otimes {t_2}^3 + 4t_2 \otimes {t_2}^4. \tag{6}$$

$\mathcal{P}^1 a_2 = {a_2}^5$ implies $\mathcal{P}^1_* t_{10} = t_2$. Define t'_{22} by $y_{12}t_{10} - {t_2}^4 t_{14}$. Then by (6) and Proposition 7, $\overline{\phi}(t'_{22}) = \Delta^*(y_{12})\phi(t_{10}) - \phi(t_2)^4\phi(t_{14}) = t'_{22} \otimes 1 + 1 \otimes t'_{22}$. On the other hand, since $\mathcal{P}^1_* y_{12}$ and $\mathcal{P}^1_* t_{14}$ are trivial, $\mathcal{P}^1_* t'_{22} = y_{12}\mathcal{P}^1_* t_{10} = y_{12}t_2 = t_{14}$. So t'_{22} is nontrivial and primitive. Put $t_{22} = t'_{22}$. Now we obtain the following equations.

$$y_{12}t_{10} = t_{22} - {t_2}^4 t_{14}, \tag{7}$$

$$\mathcal{P}^1_* t_{22} = t_{14}. \tag{8}$$

Using Proposition 7 and ${y_{12}}^5 = 0$, we can compute ${y_{12}}^4 t_{22}$, that is,

$$\begin{aligned} {y_{12}}^4 t_{22} &= {y_{12}}^4(y_{12}t_{10} + {t_2}^4 t_{14}) \\ &= {y_{12}}^5 t_{10} + {y_{12}}^4({t_2}^4 t_{14}) \\ &= {y_{12}}^4({t_2}^4 t_{14}) \end{aligned}$$

Here, since $y_{12}t_j$ $(j = 14, 26, 38)$ is primitive, there exists $\rho_j \in Z/5$ such that $y_{12}t_j = \rho_j t_{j+12}$, where $t_{50} = {t_{10}}^5$. Note that $y_{12}({t_{10}}^5) = 0$. Therefore modulo the ideal $(t_{26}, t_{38}, {t_{10}}^5)$, we have

$${y_{12}}^4({t_2}^4 t_{14}) \equiv 4{y_{12}}^3({t_2}^3 {t_{14}}^2) \equiv 12{y_{12}}^2({t_2}^2 {t_{14}}^3) \equiv 24y_{12}(t_2 {t_{14}}^4) \equiv -{t_{14}}^5.$$

But, since ${y_{12}}^4 t_{22}$ is primitive, we obtain ${y_{12}}^4 t_{22} = -{t_{14}}^5$. This means that ${y_{12}}^i t_{22}, (1 \leq i \leq 4)$, are nontrivial primitive elements. Therefore we can define the generators so that

$$t_{22+12i} = {y_{12}}^i t_{22}, (1 \leq i \leq 3). \tag{9}$$

Next we will observe ${y_{12}}^i t_{14}, (1 \leq i \leq 3)$. Since $\mathcal{P}^1_* t_{58}$ is primitive, there is $\epsilon \in \mathrm{Z}/5$ such that $\mathcal{P}^1_* t_{58} = \epsilon {t_{10}}^5$. On the other hand, from Proposition 3, $\mathcal{P}^1 a_{50} \equiv a_{58}$ (up to non zero coefficient and modulo decomposable). Dualize it, then we can see $\epsilon \neq 0$. Re-define t_{58} by $\epsilon^{-1}{y_{12}}^3 t_{22}$. We have

Proposition 10

$$y_{12}{}^3 t_{22} = \epsilon t_{58}, \quad (10)$$

$$\mathcal{P}^1_* t_{58} = t_{10}{}^5. \quad (11)$$

From this, $y_{12}{}^3 t_{14} = y_{12}{}^3 \mathcal{P}^1_* t_{22} = \mathcal{P}^1_*(y_{12}{}^3 t_{22}) = \mathcal{P}^1_*(\epsilon t_{58}) = \epsilon t_{10}{}^5$. So we can fix

$$t_{14+12i} = y_{12}{}^i t_{14}, \ (1 \le i \le 2). \quad (12)$$

By $\mathcal{P}^1_*(y_{12}{}^i t_{2k}) = y_{12}{}^i \mathcal{P}^1_* t_{2k}$, $\mathcal{P}^1_*$ is determined on all t_{2k}.

We summarize the results.

Therorem 11 *In Therem 4, we can chose the generators satisfying the following table :*

t_{2j}	t_2	t_{10}	t_{14}	t_{22}	t_{26}	t_{34}	t_{38}	t_{46}	t_{58}
$y_{12}t_{2j}$	t_{14}	$t_{22} - t_2{}^4 t_{14}$	t_{26}	t_{34}	t_{38}	t_{46}	$\epsilon t_{10}{}^5$	ϵt_{58}	$-\epsilon^{-1} t_{14}{}^5$
$\mathcal{P}^1_* t_{2j}$	0	t_2	0	t_{14}	0	t_{26}	0	t_{38}	$t_{10}{}^5$

All $t_{2k}, (k \neq 5)$ *are primitive and*

$$\overline{\phi}(t_{10}) = 4t_2{}^4 \otimes t_2 + 3t_2{}^3 \otimes t_2{}^2 + 3t_2{}^2 \otimes t_2{}^3 + 4t_2 \otimes t_2{}^4.$$

[Proof of Theorem 6] Put $y_{2n(j)+1} = \sigma(t_{2n(j)})$,$(3 \le j \le 7)$. Theorem 6 is an immediate consequence of Theorem 1, Theorem 4 and Proposition 5 with Proposition 7. ∎

Fix the basis of $H_*(E_8; \mathrm{Z}/5)$:

$$\{\Pi_{j=1}^7 y_{2n(j)+1}{}^{\epsilon_{2n(j)+1}} y_{11}{}^{\epsilon_{11}} y_{12}{}^e | 0 \le \epsilon_i \le 1, \quad 0 \le e < 5\}.$$

Let $(\ )^*$ be the dual with respect to the above basis. We may assume that $x_{2n(j)+1} = (y_{2n(j)+1})^*$, $(2 \le j \le 7)$. Let φ be the coproduct of $H^*(E_8; \mathrm{Z}/5)$ and $\overline{\varphi}(x) = \varphi(x) - x \otimes 1 - 1 \otimes x$. Then the following theorem is easily obtained by dualizing Theorem 6.

Therorem 12 *In Theorem 1, we can chose the generators satisfying following tables:*

x_i	x_3	x_{11}	x_{12}	x_{15}	x_{23}	x_{27}	x_{35}	x_{39}	x_{47}
$\mathcal{P}^1 x_i$	x_{11}	0	0	x_{23}	0	x_{35}	0	x_{47}	0
βx_i	0	x_{12}	0	0	$x_{12}{}^2/2$	0	$x_{12}{}^3/3!$	0	$x_{12}{}^4/4!$

x_i	$\overline{\varphi} x_i$
x_{15}	$x_{12} \otimes x_3$
x_{23}	$x_{12} \otimes x_{11}$
x_{27}	$x_{12} \otimes x_{15} + {x_{12}}^2/2 \otimes x_3$
x_{35}	$x_{12} \otimes x_{23} + {x_{12}}^2/2 \otimes x_{11}$
x_{39}	$x_{12} \otimes x_{27} + {x_{12}}^2/2 \otimes x_{15} + {x_{12}}^3/3! \otimes x_3$
x_{47}	$x_{12} \otimes x_{35} + {x_{12}}^2/2 \otimes x_{23} + {x_{12}}^3/3! \otimes x_{11}$

(Remark) In [6], $x_{2n(j)+1}$, $(4 \leq j \leq 7)$, are chosen as our $2x_{27}, 2x_{35}, 3!x_{39}$ and $3!x_{47}$ respectively.

Hiroaki HAMANAKA
Department of Mathematics, Kyoto University

References

[1] S. Araki. *On the non-commutativity of Pontrjagin rings mod 3 of some compact exceptional groups.* Nagoya Math. J., **17**(1960), 225-260.

[2] A. Borel. *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes.* Amer. J. Math., **76**(1954), 273-342.

[3] R. Bott. *The space of loops on Lie group.* Michigan Math. J., **5**(1955), 35-61.

[4] H.Hamanaka. *Homology ring mod2 of free loop groups of exceptional Lie groups.* J.Math.Kyoto Univ., to appear.

[5] R.M. Kane. *The Homology of Hopf Spaces.* North-Holloand Mathematical Library **40**(1988)

[6] A. Kono. *Hopf algebra structure of simple Lie groups.* Journal of Mathematics of Kyoto University, **17-2**(1977),259-298.

[7] A. Kono. *On the cohomology mod 2 of E_8.* Journal of Mathematics of Kyoto University, **24-2**(1984),275-280

[8] A. Kono and K. Kozima. *The adjoint action of Lie group on the space of loops.* Journal of The Mathematical Society of Japan, **45** No.3(1993),495-510.

[9] T. Petrie *The weakly complex bordism of Lie groups.* Ann. Math., **88**(1968), 371-402.

[10] M. Rothenberg and N. Steenrod. *The cohomology of the classifying spaces of H-spaces.* Bull. Helv., **45**(1961), 211-264.