The Mod 3 Homology of the Space of Loops on the Exceptional Lie Groups and the Adjoint Action

Hiroaki HAMANAKA*
Department of Mathematics, Kyoto University
and
Shin-ichiro HARA
Nagaoka University of Technology.

July 10, 1997

1 Introduction

Let p be a prime number and G be a compact, connected, simply connected and simple Lie group. Let ΩG be the loop space of G. Bott showed $H_*(\Omega G; \mathrm{Z}/p)$ is a finitely generated bicommutative Hopf algebra concentrated in even degrees, and determined it for classical groups G ([1]).

Here, let G be an exceptional Lie group, that is, $G = G_2, F_4, E_6, E_7, E_8$. In [2], K.Kozima and A.Kono determined $H_*(\Omega G; \mathrm{Z}/2)$ as a Hopf algebra over $\mathcal{A}_2$, where $\mathcal{A}_p$ is the mod p Steenrod Algebra and acts on it dually.

Let $\mathrm{Ad} : G \times G \to G$ and $\mathrm{ad} : G \times \Omega G \to \Omega G$ be the adjoint actions of G on G and ΩG respectively. In [3], the cohomology maps of these adjoint actions are studied and it is shown that $H^*(ad; \mathrm{Z}/p) = H^*(p_2; \mathrm{Z}/p)$ where p_2 is the second projection if and only if $H^*(G; \mathrm{Z})$ is p-torsion free. For $p = 2, 3$ and 5, some exceptional Lie groups have p-torsions on its homology.

*Partially supported by JSPS Research Fellowships for Young Scientists.

Moreover in [8, 9] mod p homology map of ad is determined for $(G,p) = (G_2,2),(F_4,2),(E_6,2),(E_7,2)$ and $(E_8,5)$. This result is applied to compute the $\mathcal{A}_5$ module structure of $H_*(\Omega E_8;\mathrm{Z}/5)$ and $H^*(E_8;\mathrm{Z}/5)$ in [9].

For a compact and connected Lie group G, the free loop group of G is denoted by $LG(G)$, i.e. the space of free loops on G equipped with multiplication as

$$\phi \cdot \psi(t) = \phi(t) \cdot \psi(t),$$

and has ΩG as its normal subgroup. Then

$$LG(G)/\Omega G \cong G,$$

and identifying elements of G with constant maps from S^1 to G, $LG(G)$ is equal to the semi-direct product of G and ΩG. This means that the homology of $LG(G)$ is determined by the homology of G and ΩG as module and the algebra structure of $H_*(LG(G);Z/p)$ depends on $H_*(ad;Z/p)$ where

$$ad : G \times \Omega G \to \Omega G$$

is the adjoint map. Since the next diagram commutes where λ,λ' and μ are the multiplication maps of ΩG, $LG(G)$ and G respectively and ω is the composition

$$(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G\times G} \times ad \times 1_G) \circ (1_{\Omega G} \times \Delta_G \times 1_{\Omega G\times G}),$$

$$\begin{tikzcd}
\Omega G \times G \times \Omega G \times G \arrow[r, "\omega"] \arrow[d, "\cong \times \cong"] & \Omega G \times \Omega G \times G \times G \arrow[r, "\lambda \times \mu"] & \Omega G \times G \arrow[d, "\cong"] \\
LG(G) \times LG(G) \arrow[rr, "\lambda'"] & & LG(G)
\end{tikzcd}$$

we can determine directly the algebra structure of $H_*(LG(G);Z/p)$ by the knowledge of the Hopf algebra structures of $H_*(G;Z/p)$, $H_*(\Omega G;Z/p)$ and induced homology map $H_*(ad;Z/p)$. See Theorem 6.12 of [8] for detail.

In this paper we determine the Hopf algebra structure over $\mathcal{A}_3$ of the homology group $H_*(\Omega G;\mathrm{Z}/3)$ for $G = F_4, E_6, E_7$ and E_8 by using adjoint action and determine the mod 3 homology map of ad for them. The result is shown in §2.

This paper is organized as follows. We refer to the results of [4, 5, 6] for the structure of $H^*(G)$ and compute $H^*(\Omega G)$ for the lower dimensions and their cohomology operations are partially determined. This is done in §3.

In §4 we turn to their homology rings. We determine the algebra structure of $H_*(\Omega G\,;\mathrm{Z}/3)$ and we partly determine the Hopf algebra structure and cohomology operations on $H_*(\Omega G;\mathrm{Z}/3)$. Finally in §5 the homology map of the adjoint action and the rest of the Hopf algebra structure and cohomology operations are determined. The computations are completely algebraic.

2 Results

Let $G(l)$ be the compact, connected, simply connected and simple exceptional Lie group of rank l where $l = 4, 6, 7$ or 8. The exponents of $G(l)$ are the integers $n(1) < n(2) < \cdots < n(l)$ which are given by the following table :

l	$n(1)$,		$n(2)$,		...	,	$n(l)$					
4	1		5	7			11					
6	1	4	5	7	8		11					
7	1		5	7		9	11	13	17			
8	1			7			11	13	17	19	23	29

Put $E(l) = \{n(1), \cdots,\ n(l)\}$ and $\overline{\phi}(t) = \Delta_*(t) - (t \otimes 1 + 1 \otimes t)$ where Δ is the diagonal map. $\mathcal{P}^k_*$ is the dual of the Steenrod operation $\mathcal{P}^k$. Then the results are following :

Therorem 1. *As a Hopf Algebra over $\mathcal{A}_3$,*

$$H_*(\Omega G(l);\mathrm{Z}/3) \;\cong\; \begin{cases} \mathrm{Z}/3[t_{2j} | j \in E(l) \cup \{3\}]/(t_2{}^3), & \textit{if } l = 4, 6, 7 \\ \mathrm{Z}/3[t_{2j} | j \in E(8) \cup \{3, 9\}]/(t_2{}^3, t_6{}^3), & \textit{if } l = 8 \end{cases}$$

where $|t_{2j}| = 2j$.

$$\overline{\phi}(t_{2j}) = \begin{cases} 0, & \textit{if } j \neq 3, 9, \\ -t_2{}^2 \otimes t_2 - t_2 \otimes t_2{}^2, & \textit{if } j = 3, \\ t_2{}^2 t_6{}^2 \otimes t_2 + t_2 t_6{}^2 \otimes t_2{}^2 - t_6{}^2 \otimes t_6 - t_2{}^2 t_6 \otimes t_2 t_6 & \\ \quad -t_2 t_6 \otimes t_2{}^2 t_6 - t_6 \otimes t_6{}^2 + t_2{}^2 \otimes t_2 t_6{}^2 + t_2 \otimes t_2{}^2 t_6{}^2, & \textit{if } j = 9, \end{cases}$$

$$\mathcal{P}^{3^r}_* t_{2j} = 0, \quad \textit{if} \quad r \geq 3,$$

$$\mathcal{P}^9_* t_{2j} = \begin{cases} t_{22}, & \textit{if } j = 29, \\ 0, & \textit{otherwise.} \end{cases}$$

$\mathcal{P}^1_* t_{2j}$ and $\mathcal{P}^3_* t_{2j}$ are given by the following table:

t_{2j}	t_2	t_6	t_8	t_{10}	t_{14}	t_{16}	t_{18}	t_{22}	t_{26}	t_{34}	t_{38}	t_{46}	t_{58}
$\mathcal{P}^1_* t_{2j}$	0	t_2	0	0	t_{10}	0	$\epsilon t_{14} - t_2 {t_6}^2$	$\kappa {t_6}^3$	ϵt_{22}	$-\epsilon {t_{10}}^3$	ϵt_{34}	$\epsilon {t_{14}}^3$	${t_{18}}^3$
$\mathcal{P}^3_* t_{2j}$	0	0	0	0	0	0	t_6	0	t_{14}	t_{22}	$-t_{26}$	t_{34}	0

where ϵ and κ are 1 or -1.

(Remark) In Theorem 1, if t_{2j} does not exist in $H_*(\Omega G(l)\ ; \mathrm{Z}/3)$, we regard t_{2j} as 0 for such j.

Let $\mathrm{Ad} : G \times G \to G$ and $\mathrm{ad} : G \times \Omega G \to \Omega G$ be the adjoint actions of a Lie group G defined by $\mathrm{Ad}(g, h) = ghg^{-1}$ and $\mathrm{ad}(g, l)(t) = gl(t)g^{-1}$ where $g, h \in G$, $l \in \Omega G$ and $t \in [0, 1]$. These induce the homology maps

$$\begin{aligned} \mathrm{Ad}_* &: H_*(G; \mathrm{Z}/3) \otimes H_*(G; \mathrm{Z}/3) \to H_*(G; \mathrm{Z}/3) \\ \mathrm{ad}_* &: H_*(G; \mathrm{Z}/3) \otimes H_*(\Omega G\ ; \mathrm{Z}/3) \to H_*(\Omega G\ ; \mathrm{Z}/3). \end{aligned}$$

Therorem 2. *There are generators y_8 in $H_*(G(l); \mathrm{Z}/3)$ for $l = 4, 6, 7$ and y_8 and y_{20} in $H_*(E_8; \mathrm{Z}/3)$. We can choose these generators so that $ad_*(y_i \otimes t_{2j})$ $(i = 8, 20)$ is given by the following table.*

t_{2j}	$ad_*(y_8 \otimes t_{2j})$	$ad_*(y_{20} \otimes t_{2j})$	t_{2j}	$ad_*(y_8 \otimes t_{2j})$	$ad_*(y_{20} \otimes t_{2j})$
t_2	t_{10}	ϵt_{22}	t_{22}	$-{t_{10}}^3$	$-{t_{14}}^3$
t_6	$t_{14} - t_{10}{t_2}^2$	$t_{26} - \epsilon t_{22}{t_2}^2$	t_{26}	t_{34}	$-t_{46}$
t_8	t_{16}	—	t_{34}	$-{t_{14}}^3$	$\epsilon {t_{18}}^3$
t_{10}	$\kappa {t_6}^3$	—	t_{38}	$-t_{46}$	t_{58}
t_{14}	t_{22}	t_{34}	t_{46}	$-\epsilon {t_{18}}^3$	$\epsilon {t_{22}}^3$
t_{16}	$\delta {t_8}^3$	—	t_{58}	$-\epsilon {t_{22}}^3$	$-{t_{26}}^3$
t_{18}	$t_{26} + t_{10}{t_6}^2{t_2}^2 - t_{14}{t_6}^2$	$t_{38} + \epsilon t_{22}{t_6}^2{t_2}^2 - t_{26}{t_6}^2$			

where $\delta, \epsilon \in \mathrm{Z}/3\mathrm{Z}$ and $\epsilon \neq 0$. For other generators $y_i \in H_*(G(l); \mathrm{Z}/3)$, $ad_*(y_i \otimes t_{2j}) = 0$ for all j.

3 The mod 3 cohomology groups

We recall the results of [4, 5, 6] for the structure of $H^*(G(l); \mathrm{Z}/3)$ as the Hopf algebra over $\mathcal{A}_3$.

Therorem 3. *There is an isomorphism :*

$$H^*(G(l);\mathbb{Z}/3) \cong \begin{cases} \Lambda(x_{2j+1} | j \in E(l) \cup \{3\} - \{11\}) \otimes \mathbb{Z}/3[x_8]/({x_8}^3), & \text{if } l = 4, 6, 7, \\ \Lambda(x_{2j+1} | j \in E(8) \cup \{3, 9\} - \{11, 29\}) \otimes \mathbb{Z}/3[x_8, x_{20}]/({x_8}^3, {x_{20}}^3), & \\ & \text{if } l = 8, \end{cases}$$

the coproduct is given by :

x_i	$\overline{\varphi} x_i$
x_{11}	$x_8 \otimes x_3$
x_{15}	$x_8 \otimes x_7$
x_{17}	$x_8 \otimes x_9$
x_{27}	$x_8 \otimes x_{19} + x_{20} \otimes x_7$
x_{35}	$x_8 \otimes x_{27} - {x_8}^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_8 x_{20} \otimes x_7$
x_{39}	$x_{20} \otimes x_{19}$
x_{47}	$-x_8 \otimes x_{39} - x_{20} \otimes x_{27} - x_{20} x_8 \otimes x_{19} + {x_{20}}^2 \otimes x_7$
others	*0*

and the cohomology operations are determined by the following table:

x_i	x_3	x_7	x_8	x_9	x_{11}	x_{15}	x_{17}	x_{19}	x_{20}	x_{27}	x_{35}	x_{39}	x_{47}
βx_i	0	x_8	0	0	0	$-{x_8}^2$	0	x_{20}	0	$x_8 x_{20}$	$-{x_8}^2 x_{20}$	$-{x_{20}}^2$	$x_8 {x_{20}}^2$
$\mathcal{P}^1 x_i$	x_7	0	0	0	x_{15}	ϵx_{19}	0	0	0	0	ϵx_{39}	0	0
$\mathcal{P}^3 x_i$	0	x_{19}	x_{20}	0	0	x_{27}	0	0	0	$-x_{39}$	x_{47}	0	0

where ϵ is 1 or -1.

If $r > 1$ then $\mathcal{P}^{3^r} x_i = 0$.

(Remark) We consider x_i in these tables as 0 when $x_i \notin H^*$.

Recall a Serre fibration:

$$\Omega G(l) \longrightarrow * \longrightarrow G(l). \qquad \text{(A)}$$

First, we compute $H^*(\Omega G(l);\mathbb{Z}/3)$ by the Serre spectral sequence associated with the fibration (A). This spectral sequence has a Hopf algebra structure. We can proceed to compute it using degree-reason and Kudo's transgression theorem ([7]) from the previous theorem. For $j \in E(l) - \{9, 11, 29\}$, there are universally transgressive elements $a_{2j} \in H^*(\Omega G(l);\mathbb{Z}/3)$, such that $\tau a_{2j} = x_{2j+1}$. Thus we can show that for $j = 9, 11, 15, 21, 27$ and 29, there are a_{2j} such that satisfy

$$
\begin{array}{lll}
d_7(1\otimes a_{18}) & = x_7\otimes {a_2}^6, & \text{for } l=4,6,7,\\
d_{11}(1\otimes a_{30}) & = x_{11}\otimes {a_{10}}^2, & \text{for } l=4,6,7,\\
d_{15}(1\otimes a_{42}) & = x_{15}\otimes {a_{14}}^2, & \text{for } l=8,\\
d_{19}(1\otimes a_{22}) & = x_3{x_8}^2\otimes {a_2}^2, & \text{for } l=4,6,7,8,\\
d_{19}(1\otimes a_{54}) & = x_{19}\otimes {a_2}^{18}, & \text{for } l=8,\\
d_{47}(1\otimes a_{58}) & = x_7{x_{20}}^2\otimes {a_2}^6, & \text{for } l=8.
\end{array}
$$

${a_{2j}}'s$ are generators of the cohomology group in the low dimensions. The results are the following:

Proposition 4. *For the dimensions less than $2n(l)+2$, the next isomorphism holds :*

$$
H^*(\Omega G(l);\mathrm{Z}/3)\cong\begin{cases}
\mathrm{Z}/3[a_{2j}|j\in E(l)\cup\{9\}]/({a_2}^9), & \textit{if } l=4,6,\\
\mathrm{Z}/3[a_{2j}|j\in E(7)\cup\{15\}]/({a_{10}}^3), & \textit{if } l=7,\\
\mathrm{Z}/3[a_{2j}|j\in E(8)\cup\{21,27\}]/({a_2}^{27},{a_{14}}^3), & \textit{if } l=8.
\end{cases}
$$

Now we start to determine the cohomology operations and the coproducts on a_{2j}.

Therorem 5. *For $j\in E(l)-\{9,11,29\}$ $a_{2j}\in H^*(\Omega G(l);\mathrm{Z}/3)$ is primitive and cohomology operations are determined by*

a_{2j}	a_2	a_8	a_{10}	a_{14}	a_{16}	a_{26}	a_{34}	a_{38}	a_{46}
$\mathcal{P}^1 a_{2j}$	${a_2}^3$	0	a_{14}	$\epsilon {a_2}^9$	0	0	ϵa_{38}	0	0
$\mathcal{P}^3 a_{2j}$	0	0	0	a_{26}	0	$-a_{38}$	a_{46}	0	0

If $r>1$ then $\mathcal{P}^{3^r}a_{2j}=0$.

Proof. For $j\in E(l)-\{9,11,29\}$, a_{2j} is transgressive, therefore $\mathcal{P}^i a_{2j}=\mathcal{P}^i\sigma x_{2j+1}=\sigma\mathcal{P}^i x_{2j+1}$. Thus this can be determined by Theorem 3. ∎

For the investigation of a_{2j} which is not transgressive we start from the following theorem. In the next theorem, ψ means the coproduct of $H^*(\Omega G;\mathrm{Z}/3)$ and we set $\overline{\psi}(a)=\psi(a)-(a\otimes 1+1\otimes a)$.

Therorem 6. *For $j = 9, 15, 21, 27$, $\overline{\psi} a_{2j}$ is given by the following formula:*

$$\overline{\psi} a_{2j} = \begin{cases} {a_2}^3 \otimes {a_2}^6 + {a_2}^6 \otimes {a_2}^3, & \textit{if } j = 9, \\ a_{10} \otimes {a_{10}}^2 + {a_{10}}^2 \otimes a_{10}, & \textit{if } j = 15, \\ a_{14} \otimes {a_{14}}^2 + {a_{14}}^2 \otimes a_{14}, & \textit{if } j = 21, \\ {a_2}^9 \otimes {a_2}^{18} + {a_2}^{18} \otimes {a_2}^9, & \textit{if } j = 27. \end{cases}$$

Proof. To begin with, we investigate the element a_{18}. Let a_2' be the generator of $H^2(\Omega F_4; \mathrm{Z})$. $H^*(\Omega F_4; \mathrm{Z})$ has no torsion and is a commutative Hopf algebra over Z. Since ${a_2}^9 = 0$, there is a_{18}' such that ${a_2'}^9 = 3a_{18}'$ and $\rho a_{18}' \neq 0$, where ρ is modulo 3 reduction. Then we can choose a_{18} as $\rho a_{18}'$. The coproduct of a_{18}' is computed as follows:

$$\begin{aligned} \psi a_{18}' &= 1/3 \psi {a_2'}^9 \\ &= 1/3(1 \otimes a_2' + a_2' \otimes 1)^9 \\ &\equiv a_{18}' \otimes 1 + {a_2'}^3 \otimes {a_2'}^6 + {a_2'}^6 \otimes {a_2'}^3 + 1 \otimes a_{18}' \ (\textit{mod } 3). \end{aligned}$$

Thus $\overline{\psi} a_{18} = {a_2}^3 \otimes {a_2}^6 + {a_2}^6 \otimes {a_2}^3$ is shown.

Consider the inclusion $\iota : F_4 \longrightarrow E_7$, we chose $a_{18} \in H^*(\Omega E_7; \mathrm{Z}/3)$ so as to satisfy $(\Omega\iota)^* a_{18} = a_{18}$. Because $(\Omega\iota)^*$ is injective for degrees less than 18, $\overline{\psi} a_{18} = {a_2}^3 \otimes {a_2}^6 + {a_2}^6 \otimes {a_2}^3$ is shown again for this a_{18}. And in the similar way we put $a_{30} = 1/3{a_{10}}^3, a_{42} = 1/3{a_{14}}^3$ and $a_{54} = 1/3{a_2}^{27}$ and obtain the coproduct formulas of the statement. ∎

We remark that we can assume that a_{22} and a_{58} are primitive.

Therorem 7. *In Proposition 4 we have that $\mathcal{P}^1 a_{18} = \pm a_{22}$.*

Let $\widetilde{G}(l)$ be the 3–connected cover of $G(l)$ and

$$\begin{array}{ccccccc} \Omega\widetilde{G}(l) & \longrightarrow & * & \longrightarrow & \widetilde{G}(l) & & (\mathrm{B}) \\ \widetilde{G}(l) & \xrightarrow{p} & G(l) & \xrightarrow{i} & K(\mathrm{Z}, 3) & & (\mathrm{C}) \\ \Omega\widetilde{G}(l) & \xrightarrow{\Omega p} & \Omega G(l) & \xrightarrow{\Omega i} & K(\mathrm{Z}, 2) & & (\mathrm{D}) \end{array}$$

be Serre fibrations. To prove Theorem 7 we have to compute $H^*(\Omega\widetilde{G}; \mathrm{Z}/3)$ and $H^*(\widetilde{G}; \mathrm{Z}/3)$.

Let $\tilde{a}_{2j}$ be $(\Omega p)^* a_{2j}$, for $j \neq 1$. Using the Serre spectral sequence associated with the fibration (D), one can easily show that there are generators $\tilde{a}_{17} \in H^{17}$ for $l = 4, 6$, and $\tilde{a}_{53} \in H^{53}$ for $l = 8$. We have the following proposition. Let denote $E(l) - \{1\}$ as $\tilde{E}(l)$.

Proposition 8. *For the dimensions less than $2n(l)+2$, the next isomorphism holds :*

$$H^*(\Omega\widetilde{G}(l);\mathrm{Z}/3) \cong \begin{cases} \mathrm{Z}/3[\tilde{a}_{2j}|j\in\tilde{E}(l)\cup\{9\}]\otimes\Lambda(\tilde{a}_{17}), & \text{if } l=4,6,\\ \mathrm{Z}/3[\tilde{a}_{2j}|j\in\tilde{E}(7)\cup\{15\}]/(\tilde{a}_{10}^3), & \text{if } l=7,\\ \mathrm{Z}/3[\tilde{a}_{2j}|j\in\tilde{E}(8)\cup\{21,27\}]/(\tilde{a}_{14}^3)\otimes\Lambda(\tilde{a}_{53}), & \text{if } l=8.\end{cases}$$

By computing the Serre spectral sequence associated with (B), it is easy to see $\tilde{a}_{2j}, (j\neq 15,21)$ is universally transgressive. Let $\tilde{x}_{i+1}$ be $\tau\tilde{a}_i$. Then we have the following:

Proposition 9. *For the dimensions less than $2n(l)+2$, the next isomorphism holds :*

$$H^*(\widetilde{G}(l);\mathrm{Z}/3) \cong \begin{cases} \Lambda(\tilde{x}_{2j+1}|j\in\tilde{E}(l)\cup\{9\})\otimes\mathrm{Z}/3[\tilde{x}_{18}], & \text{if } l=4,6,\\ \Lambda(\tilde{x}_{2j+1}|j\in\tilde{E}(7)), & \text{if } l=7,\\ \Lambda(\tilde{x}_{2j+1}|j\in\tilde{E}(8)\cup\{27\})\otimes\mathrm{Z}/3[\tilde{x}_{54}], & \text{if } l=8.\end{cases}$$

Proof of Theorem 7. It is possible to show that $\mathcal{P}^1a_{18}$ is not zero as follows. Let σ' denotes the cohomology suspension associated to the fibration (C) for $l=4$. It is easy to see $\tilde{x}_{19}=\sigma'\beta\mathcal{P}^3\mathcal{P}^1u_3$ and $\tilde{x}_{23}=\sigma'(\beta\mathcal{P}^1u_3)^3$, where u_3 is the generator of $H^3(\mathrm{K}(\mathrm{Z},3);\mathrm{Z}/3)$. So we get $\mathcal{P}^1\tilde{x}_{19}=\sigma'\mathcal{P}^1\beta\mathcal{P}^3\mathcal{P}^1u_3=\sigma'\mathcal{P}^4\beta\mathcal{P}^1u_3=\sigma'(\beta\mathcal{P}^1u_3)^3=\tilde{x}_{23}$, and from this, we have $(\Omega p)^*\mathcal{P}^1a_{18}=\mathcal{P}^1(\Omega p)^*a_{18}=\mathcal{P}^1\tilde{a}_{18}=\mathcal{P}^1\sigma\tilde{x}_{19}=\sigma\mathcal{P}^1\tilde{x}_{19}=\sigma\tilde{x}_{23}=\tilde{a}_{22}$, where σ is the cohomology suspension associated to (B). Thus $\mathcal{P}^1a_{18}\neq 0$. We fix a_{22} as $\mathcal{P}^1a_{18}$. ∎

4 Homology groups

Therorem 10. *The homology ring of $\Omega G(l)$ is*

$$H_*(\Omega G(l);\mathrm{Z}/3) \;\cong\; \begin{cases} \mathrm{Z}/3[t_{2j}|j\in E(l)\cup\{3\}]/({t_2}^3), & \text{if } l=4,6,7\\ \mathrm{Z}/3[t_{2j}|j\in E(8)\cup\{3,9\}]/({t_2}^3,{t_6}^3), & \text{if } l=8.\end{cases} \tag{1}$$

where $|t_{2j}|=2j$. The coproduct is given by

$$\overline{\phi}(t_{2j}) \;=\; \begin{cases} 0, & \text{if } j\neq 3,9,11,29,\\ -{t_2}^2\otimes t_2-t_2\otimes {t_2}^2, & \text{if } j=3,\\ {t_2}^2{t_6}^2\otimes t_2+t_2{t_6}^2\otimes {t_2}^2-{t_6}^2\otimes t_6-{t_2}^2t_6\otimes t_2t_6 & \\ \quad -t_2t_6\otimes {t_2}^2t_6-t_6\otimes {t_6}^2+{t_2}^2\otimes t_2{t_6}^2+t_2\otimes {t_2}^2{t_6}^2, & \text{if } j=9.\end{cases}$$

Proof. Let t_{2j} be the dual element of $a_{2j} \in H_*(\Omega G; \mathrm{Z}/3)$ as to the monomial basis for $j \in E(l) - \{9\}$ and t_6, t_{18} be the dual element of ${a_2}^3, {a_2}^9$, respectively. It is easy to see ${t_2}^3 = {t_6}^3 = 0$ and to show the coproduct formula for t_6 and t_{18}. Thus we can say that statement (1) is true for $* < 2n(l) + 2$.

Now it is possible to show that there is no truncation in $H_*(\Omega G; \mathrm{Z}/3)$ other than the parts generated by t_2 and t_6 and that (1) holds for all dimensions. Since $H_*(\Omega G(l); \mathrm{Z}/3)$ is the even degree concentrated commutative Hopf algebra, we may suppose

$$H_*(\Omega G(l); \mathrm{Z}/3) = \mathrm{Z}/3[u_i | i \in I] \otimes \mathrm{Z}/3[v_j | j \in J]/({v_j}^{3^{r_j}} | j \in J).$$

Consider an Eilenberg - Moore spectral sequence :

$$E_2 = \mathrm{Ext}_{H_*(\Omega G(l) : \mathrm{Z}/3)}(\mathrm{Z}/3, \mathrm{Z}/3) \Longrightarrow E_\infty = \mathcal{G}r(H^*(G(l); \mathrm{Z}/3)).$$

Since $E_2 = \Lambda(su_i | i \in I) \otimes \Lambda(sv_j | j \in J) \otimes \mathrm{Z}/3[\theta v_j | j \in J]$, where $\deg su_i = (1, |u_i|)$, $\deg sv_j = (1, |v_j|)$, and $\deg \theta v_j = (2, 3^{r_j}|v_j|)$, the essential differentials have the forms : $d_r su_i = (\theta v_j)^{3^{k_j}}$ $(k_j \geq 1)$ or $d_r sv_j = (\theta v_{j'})^{3^{l_j}}$ $(l_j \geq 1)$. Because $H^*(G(l); \mathrm{Z}/3)$ is a finite dimensional vector space, one can easily show

$$E_\infty = \Lambda(su_i | i \in I') \otimes \Lambda(sv_j | j \in J') \otimes \mathrm{Z}/3[\theta v_j | j \in J]/((\theta v_j)^{3^{m_j}} | j \in J), \quad (I' \subset I, J' \subset J)$$

and $|I'| + |J'| = |I|$. Here the total dimension of E_∞ is $2^{|I'|+|J'|} 3^{\sum_{j \in J} m_j}$, $(m_j \geq 1)$ and the total dimension of $H^*(G(l); \mathrm{Z}/3)$ is $2^{|E(l)|} 3^{f(l)}$ where $f(l) = 1$ for $l = 4, 6, 7$ and $f(l) = 2$ for $l = 8$. Thus the indices J of the truncation part satisfy that $|J| \leq f(l)$ and $|I| = |E(l)|$. This means that the truncation parts of $H_*(\Omega G; \mathrm{Z}/3)$ is generated by only t_2 and t_6.

Therefore $H_*(\Omega G(l)\ ; \mathrm{Z}/3)$ has the form

$$\begin{array}{ll} \mathrm{Z}/3[u_i | i \in I] \otimes \mathrm{Z}/3[t_2]/({t_2}^3) & \text{for } l = 4, 6, 7 \text{ and} \\ \mathrm{Z}/3[u_i | i \in I] \otimes \mathrm{Z}/3[t_2, t_6]/({t_2}^3, {t_6}^3) & \text{for } l = 8. \end{array}$$

Also Theorem 5 means that for $j \in E(l) - \{9\}$ t_{2j} is primitive and indecomposable and t_6, t_{18} are indecomposable. Thus

$$\begin{array}{ll} \{t_{2j} | j \in \tilde{E}(l)\} \cup \{t_6\} \subset \{u_i | i \in I\} & \text{for } l = 4, 6, 7 \text{ and} \\ \{t_{2j} | j \in \tilde{E}(l)\} \cup \{t_{18}\} \subset \{u_i | i \in I\} & \text{for } l = 8. \end{array}$$

Since $|I| = |E(l)|$, the theorem is proved. ∎

Dualizing the result of Theorem 5 and Theorem 7, we obtain the statement of Theorem 1 except for $\mathcal{P}^1_* t_{26}, \mathcal{P}^1_* t_{34}, \mathcal{P}^3_* t_{34}, \mathcal{P}^1_* t_{46}, \mathcal{P}^1_* t_{58}$ and $\mathcal{P}^9_* t_{58}$. To determine these operations, we use the adjoint action of $H_*(G(l); \mathrm{Z}/3)$ on $H_*(\Omega G(l)\ ; \mathrm{Z}/3)$ which is introduced in the next section.

(Remark)The computation of dualizing the result of Theorem 5 and Theorem 7 is not difficult except for $\mathcal{P}^1_* t_{18}$, because $\mathcal{P}^n_* t$ is primitive if t is primitive. Moreover, it is easily shown

$$\overline{\phi}(\mathcal{P}^1_* t_{18}) = \mathcal{P}^1_* \overline{\phi}(t_{18}) = \overline{\phi}(-t_2 {t_6}^2)$$

and this shows $\mathcal{P}^1_* t_{18} = -t_2 {t_6}^2$ modulo primitive elements. By Theorem 5 we can see $\mathcal{P}^1 a_{14} = \epsilon {a_2}^9$ and this shows that $\mathcal{P}^1_* t_{18} = \epsilon t_{14} - t_2 {t_6}^2$.

5 Adjoint action

Put $y * y' = \mathrm{Ad}_*(y \otimes y')$ and $y * t = \mathrm{ad}_*(y \otimes t)$ where $y, y' \in H_*(G; \mathrm{Z}/3)$ and $t \in H_*(\Omega G\ ; \mathrm{Z}/3)$. Following are the dual result of [3]. Also see [9].

Therorem 11. *For y, y', $y'' \in H_*(G; \mathrm{Z}/3)$ and t, $t' \in H_*(\Omega G\ ; \mathrm{Z}/3)$*

(i) $1 * y = y$, $1 * t = t$.

(ii) $y * 1 = 0$, *if* $|y| > 0$, *whether* $1 \in H_*(G; \mathrm{Z}/3)$ *or* $1 \in H_*(\Omega G\ ; \mathrm{Z}/3)$.

(iii) $(yy') * t = y * (y' * t)$.

(iv) $y * (tt') = \sum (-1)^{|y''||t|} (y' * t)(y'' * t')$ *where* $\Delta_* y = \sum y' \otimes y''$.

(v) $\sigma(y * t) = y * \sigma(t)$ *where σ is the homology suspension.*

(vi) $\mathcal{P}^n_*(y * t) = \sum_i (\mathcal{P}^i_* y) * (\mathcal{P}^{n-i}_* t)$.
$\mathcal{P}^n_*(y * y') = \sum_i (\mathcal{P}^i_* y) * (\mathcal{P}^{n-i}_* y')$.

(vii)

$$\begin{aligned} \Delta_*(y * t) &= (\Delta_* y) * (\Delta_* t) \\ &= \sum (-1)^{|y''||t'|} (y' * t') \otimes (y'' * t'') \end{aligned}$$

where $\Delta_* y = \sum y' \otimes y''$ *and* $\Delta_* t = \sum t' \otimes t''$.
And $\overline{\Delta}_*(y * t) = (\Delta_* y) * (\overline{\Delta}_* t)$.

(viii) *If t is primitive then $y * t$ is primitive.*

Also the result of [3] implies the following theorem. See [8].

Therorem 12. *We set a submodule A of $H_*(G;\mathrm{Z}/3)$ as*

$$\begin{array}{rcll} A &=& \mathrm{Z}/3[y_8]/({y_8}^3) & \text{for } G=F_4,E_6,E_7 \text{ and} \\ A &=& \mathrm{Z}/3[y_8,y_{20}]/({y_8}^3,{y_{20}}^3) & \text{for } G=E_8 \end{array}$$

where y_{2i} is the dual of x_{2i} with respect to the monomial basis. Then there exists a retraction $p: H_*(G;\mathrm{Z}/3) \to A$ and the following diagram commutes.

$$\begin{tikzcd} H_*(G;\mathrm{Z}/3)\otimes H_*(\Omega G\ ;\mathrm{Z}/3) \arrow[r, "ad_*"] \arrow[d, "p\otimes 1"] & H_*(\Omega G\ ;\mathrm{Z}/3) \\ A\otimes H_*(\Omega G\ ;\mathrm{Z}/3) \arrow[ur, "ad_*"] & \end{tikzcd}$$

(Remark)By Theorem 3 we can see $\mathcal{P}^3_* y_{20} = y_8$.

Since Ad_* is agreed with the composition $\mu_*\circ(1\otimes\mu_*)\circ(1\otimes 1\otimes\iota_*)\circ(1\otimes T)\circ(\Delta_*\otimes 1)$ where μ is the multiplication of $G(l)$ and ι is the inverse map, the next theorem follows. See [9].

Therorem 13. *Let $y, y' \in H_*(G)$. If y is primitive,*

$$y * y' = [y, y']$$

where $[y,y'] = yy' - (-1)^{|y||y'|}y'y$.

Now we give the proof of Theorem 2 and finish the proof of Theorem 1. Let y_i be the dual element of $x_i \in H^*(G(l))$ as to the monomial basis. By Theorem 3 and Theorem 13 we see that for $j \in E(l)\cup\{3,9\}-\{11,29\}$

$$y_8 * y_{2j+1} = \begin{cases} y_{2j+9} & \text{for } j=1,3,4,9,13, \\ -y_{2j+9} & \text{for } j=19, \\ 0 & \text{others} \end{cases}$$

and

$$y_{20} * y_{2j+1} = \begin{cases} y_{2j+21} & \text{for } j = 3, 7, 9, \\ -y_{2j+21} & \text{for } j = 13\ , \\ 0 & \text{others}\ . \end{cases}$$

Since $\sigma t_{2j} = y_{2j+1}$ for $j \in E(l) \cup \{3, 9\} - \{11, 29\}$, Theorem 11 (v) implies

$$\begin{array}{ll} \sigma(y_8 * t_{2j}) \neq 0 & \text{for } j = 1, 3, 4, 9, 13, 19, \\ \sigma(y_{20} * t_{2j}) \neq 0 & \text{for } j = 3, 7, 9, 13. \end{array} \tag{2}$$

Then the equations

$$y_8 * t_2 = t_{10}, \tag{3}$$

$$y_8 * t_8 = t_{16}, \tag{4}$$

$$y_8 * t_{26} = t_{34}, \tag{5}$$

$$y_8 * t_{38} = -t_{46}, \tag{6}$$

$$y_{20} * t_{14} = t_{34}, \tag{7}$$

$$y_{20} * t_{26} = -t_{46} \tag{8}$$

are shown by Theorem 11 (viii). Moreover (2) implies

$$y_8 * t_6 \equiv t_{14}, \tag{9}$$

$$y_8 * t_{18} \equiv t_{26}, \tag{10}$$

$$y_{20} * t_6 \equiv t_{26}, \tag{11}$$

$$y_{20} * t_{18} \equiv t_{38} \tag{12}$$

modulo decomposable elements. Since

$$\begin{aligned} \overline{\phi}(y_8 * t_6) &= -(y_8 * t_2) \otimes {t_2}^2 - (y_8 * {t_2}^2) \otimes t_2 - t_2 \otimes (y_8 * {t_2}^2) - {t_2}^2 \otimes (y_8 * t_2) \\ &= \overline{\phi}(-t_{10}{t_2}^2), \end{aligned}$$

one can see that $y_8 * t_6 \equiv -t_{10}{t_2}^2$ mod primitive elements. By this and (9), we have

$$y_8 * t_6 = t_{14} - t_{10}{t_2}^2. \tag{13}$$

The equations

$$y_8 * t_{18} = t_{26} + t_{10}{t_2}^2{t_6}^2 - t_{14}{t_6}^2, \tag{14}$$

$$y_{20} * t_6 = t_{26} - (y_{20} * t_2){t_2}^2, \tag{15}$$

$$y_{20} * t_{18} = t_{38} - (y_{20} * t_6){t_6}^2 \tag{16}$$

are shown in the similar way.

By the equation (13), we can compute ${y_8}^3 \otimes t_6$ as

$$\begin{aligned} {y_8}^3 * t_6 &= {y_8}^2 * (t_{14} - t_{10}{t_2}^2) \\ &= {y_8}^2 * t_{14} + {t_{10}}^3. \end{aligned}$$

Since ${y_8}^3 = 0$, ${y_8}^2 * t_{14} = -{t_{10}}^3$ and this means $y_8 * t_{14}$ is a non-zero primitive indecomposable element. We redefine t_{22} as

$$t_{22} = y_8 * t_{14}. \tag{17}$$

Then we have

$$y_8 * t_{22} = -{t_{10}}^3.$$

By Theorem 7 we can set $\mathcal{P}^1_* t_{22} = \kappa {t_6}^3$ where $\kappa = \pm 1$. Since $\mathcal{P}^1_* t_{22} = \mathcal{P}^1_*(y_8 * t_{14}) = y_8 * t_{10}$, we have

$$y_8 * t_{10} = \kappa {t_6}^3.$$

By the similar manner, we can compute ${y_8}^3 * t_{18}$ and obtain ${y_8}^2 * t_{26} = -{t_{14}}^3$. Therefore

$$y_8 * t_{34} = {y_8}^2 * t_{26} = -{t_{14}}^3. \tag{18}$$

Because t_{16} and t_{46} are primitive, we can set

$$\begin{aligned} y_8 * t_{16} &= \rho_2 {t_8}^3, && (19) \\ y_8 * t_{46} &= \rho_3 {t_{18}}^3. && (20) \end{aligned}$$

Operate $\mathcal{P}^3_*$ to (20) to obtain

$$y_8 * t_{34} = \mathcal{P}^3_*(y_8 * t_{46}) = \rho_3 \mathcal{P}^3_*({t_{18}}^3) = \rho_3 \epsilon {t_{14}}^3.$$

Thus by (18), we conclude that $\rho_3 = -\epsilon$. $y_8 * t_{58}$ will be determined after the determination of $y_{20} * t_{58}$.

Here we apply $\mathcal{P}^1_*$ on (5), (6) and (14), $\mathcal{P}^3_*$ on (5) to see

$$\begin{aligned} \mathcal{P}^1_* t_{26} &= \mathcal{P}^1_*(y_8 * t_{18} - t_{10}{t_6}^2{t_2}^2 + t_{14}{t_6}^2) \\ &= \epsilon y_8 * t_{14} = \epsilon t_{22}, \\ \mathcal{P}^1_* t_{34} &= \mathcal{P}^1_*(y_8 * t_{26}) = \epsilon y_8 * t_{22} = -\epsilon {t_{10}}^3, \\ \mathcal{P}^1_* t_{46} &= -\mathcal{P}^1_*(y_8 * t_{38}) = -\epsilon y_8 * t_{34} = \epsilon {t_{14}}^3, \\ \mathcal{P}^3_* t_{34} &= \mathcal{P}^3_*(y_8 * t_{26}) = y_8 * t_{14} = t_{22}. \end{aligned}$$

Next we compute $y_{20} * t_{2i}$. First we apply $\mathcal{P}^1_*$ to (15) to obtain

$$y_{20} * t_2 = \mathcal{P}^1_*(y_{20} * t_6) = \mathcal{P}^1_*(t_{26} - (y_{20} * t_2){t_2}^2) = \epsilon t_{22}.$$

From this, (15) and (16) imply that

$$\begin{aligned} y_{20} * t_2 &= \epsilon t_{22}, \\ y_{20} * t_6 &= t_{26} - \epsilon t_{22}{t_2}^2, \\ y_{20} * t_{18} &= t_{38} + \epsilon t_{22}{t_6}^2{t_2}^2 - t_{26}{t_6}^2. \end{aligned}$$

${y_{20}}^3 * t_6$ is computed as

$$\begin{aligned} 0 = {y_{20}}^3 * t_6 &= {y_{20}}^2 * (y_{20} * t_6) \\ &= {y_{20}}^2 * (t_{26} - \epsilon t_{22}{t_2}^2) \\ &= {y_{20}}^2 * t_{26} + \epsilon {t_{22}}^3. \end{aligned}$$

Thus $y_{20} * t_{46} = -{y_{20}}^2 * t_{26} = \epsilon {t_{22}}^3$.

The similar computation of ${y_{20}}^3 * t_{18}$ implies

$${y_{20}}^2 * t_{38} = -{t_{26}}^3.$$

Thus $y_{20} * t_{38}$ is a non zero primitive indecomposable element and we redefine t_{58} as $y_{20} * t_{38}$. Hence

$$\begin{aligned} y_{20} * t_{38} &= t_{58}, && (21) \\ y_{20} * t_{58} &= -{t_{26}}^3. && (22) \end{aligned}$$

By applying $\mathcal{P}^3_*$ to (22), we have

$$y_8 * t_{58} = \mathcal{P}^3_*(y_{20} * t_{58}) = -\mathcal{P}^3_*({t_{26}}^3) = -\epsilon {t_{22}}^3.$$

We obtain also

$$y_{20} * t_{22} = \epsilon \mathcal{P}^1_*(y_{20} * t_{26}) = -\epsilon \mathcal{P}^1_* t_{46} = -{t_{14}}^3$$

by applying $\mathcal{P}^1_*$ to (8).

Since t_{34} is primitive, we can set $y_{20} * t_{34} = \rho_4 {t_{18}}^3$ ($\rho_4 \in \mathrm{Z}/3$). Operating $\mathcal{P}^3_*$ to the both sides of this equation, $\rho_4 \epsilon {t_{14}}^3$ is computed as follows:

$$\begin{aligned} \rho_4 \epsilon {t_{14}}^3 &= \rho_4 \mathcal{P}^3_*({t_{18}}^3) \\ &= \mathcal{P}^3_*(y_{20} * t_{34}) \\ &= y_8 * t_{34} + y_{20} * t_{22} \\ &= {t_{14}}^3. \end{aligned}$$

So $y_{20} * t_{34} = \epsilon {t_{18}}^3$ is shown. Now ad_* is determined except for $y_8 * t_{16}$.

Finally we operate $\mathcal{P}^1_*$ to (21) and $\mathcal{P}^9_*$ to (22) and see

$$\mathcal{P}^1_* t_{58} = \mathcal{P}^1_*(y_{20} * t_{38}) = y_{20} * (\mathcal{P}^1_* t_{38}) = \epsilon y_{20} * t_{34} = {t_{18}}^3,$$

$$y_{20} * (\mathcal{P}^9_* t_{58}) = \mathcal{P}^9_*(y_{20} * t_{58}) = -\mathcal{P}^9_*({t_{26}}^3) = -{t_{14}}^3.$$

These equations imply that

$$\mathcal{P}^1_* t_{58} = {t_{18}}^3, \mathcal{P}^9_* t_{58} = t_{22}.$$

This completes the proof of Theorem 1.

Hiroaki HAMANAKA
Department of Mathematics, Kyoto University
Shin-ichro HARA
Nagaoka University of Technology

References

[1] R. Bott. *The space of loops on Lie group.* Michigan Math. J.**5**(1955), 35-61.

[2] A. Kono and K. Kozima. *The mod 2 homology of the space of loops on the exceptional Lie group.* Proceedings of th Royal Society of Edinburgh, **112A**(1989), 187-202.

[3] — —. *The adjoint action of Lie group on the space of loops.* Journal of The Mathematical Society of Japan, **45** No.3(1993),495-510.

[4] S. Araki. *On the non-commutativity of Pontrjagin rings mod 3 of some compact exceptional groups.* Nagoya Math.J, **17**(1960), 225-260.

[5] A. Kono and M. Mimura. *On the cohomology mod 3 of the classifying space of the compact, 1-connected exceptional Lie group E_6.* preprint series of Aarhus University, 1975.

[6] H. Toda. *Cohomology of the classifying space of exceptional Lie groups.* Conference on manifolds, Tokyo, (1973), 265-271.

[7] T. Kudo. *A transgression theorem.* Memoirs of the Faculty of Science, Kyusyu University, Ser.A, Vol.9, No.2, (1956),79-81.

[8] H. Hamanaka. *Homology ring mod 2 of free loop groups of exceptional Lie groups* J. Math. Kyoto Univ, To appear.

[9] H. Hamanaka, S. Hara and A. Kono. *Adjoint action on the modulo 5 homology groups of E_8 and ΩE_8* To appear.