HOMOLOGY RING MOD 2 OF FREE LOOP GROUPS OF EXCEPTIONAL LIE GROUPS

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1. Introduction

Assume G is a compact, connected, simply connected Lie group. The space of free loops on G is called $LG\,(G)$ the free loop group of G, whose multiplication is defined as

$$\varphi \cdot \psi(t) = \varphi(t) \cdot \psi(t).$$

Let ΩG be the space of based loops on G, whose base point is the unit e. Then $LG\,(G)$ has ΩG as its normal subgroup and

$$LG\,(G)\,/\Omega G \;\cong G.$$

Identifying elements of G with constant maps from S^1 to G, $LG\,(G)$ is equal to the semidirect product of G and ΩG . Thus the mod p homology of $LG\,(G)$ is determined by the mod p homology of G and ΩG and the algebra structure of $\mathrm{H}^*(LG\,(G)\,; \mathbf{Z}/p\mathbf{Z})$ depends on $\mathrm{H}^*(\mathrm{ad}; \mathbf{Z}/p\mathbf{Z})$ where

$$\mathrm{ad}\; : G \times \Omega G \;\to \Omega G$$

is the adjoint map.

The purpose of this paper is to determine $\mathrm{H}_*(\mathrm{ad}; \mathbf{Z}/2\mathbf{Z})$ for the exceptional Lie goups $G = G_2, F_4, E_6$ and E_7. And at the same time, using the Hopf algebra structures of $\mathrm{H}_*(\Omega E_6\,; \mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_*(\Omega E_7\,; \mathbf{Z}/2\mathbf{Z})$, we could determine the $\mathcal{A}_2^*$ module structure of $\mathrm{H}_*(\Omega G\,; \mathbf{Z}/2\mathbf{Z})$. Moreover some mistakes was detected in the result about Hopf structure of $\mathrm{H}_*(\Omega E_6\,; \mathbf{Z}/2\mathbf{Z})$ of [5] and we offer the modified result. The main result is showed in Theorem 4.1, 4.4 and 5.2.

This paper is organized as follows. In §2 we refer to the result of the algebra structure of $\mathrm{H}^*(G; \mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_*(\Omega G\,; \mathbf{Z}/2\mathbf{Z})$. And in §3 we introduce the adjoint action and observe its property and in §4, §5 the induced homomorphism from adjoint action of G_2, F_4, E_6 and E_7 is determined. Finally in §6 we give the method to compute the Pontrjagin ring of $LG\,(G)$ and show the case of G_2.

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2. $\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z})$ AND $\mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$

We refer to the result of [1] and [2] about $\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z})$ for $G = G_2, F_4, E_6, E_7$.

Theorem 2.1.

$$\begin{aligned}
\mathrm{H}^*(G_2;\mathbf{Z}/2\mathbf{Z}) &= \mathbf{Z}/2\mathbf{Z}[x_3]/(x_3^4)\otimes \bigwedge(x_5),\\
\mathrm{H}^*(F_4;\mathbf{Z}/2\mathbf{Z}) &= \mathbf{Z}/2\mathbf{Z}[x_3]/(x_3^4)\otimes \bigwedge(x_5,x_{15},x_{23}),\\
\mathrm{H}^*(E_6;\mathbf{Z}/2\mathbf{Z}) &= \mathbf{Z}/2\mathbf{Z}[x_3]/(x_3^4)\otimes \bigwedge(x_5,x_9,x_{15},x_{17},x_{23}),\\
\mathrm{H}^*(E_7;\mathbf{Z}/2\mathbf{Z}) &= \mathbf{Z}/2\mathbf{Z}[x_3,x_5,x_9]/(x_3^4,x_5^4,x_9^4)\otimes \bigwedge(x_{15},x_{17},x_{23},x_{27})
\end{aligned}$$

where x_i is a generator of degree i. Moreover there are homomorphisms

$$G_2 \to F_4 \to E_6 \to E_7$$

whose induced homomorphism map x_i into x_i in the cohomology of any smaller group.

Theorem 2.2. *The x_i in Theorem 2.1 can be chosen so as to satisfy*

$$\begin{aligned}
x_5 &= \mathrm{Sq}^2 x_3,\\
x_9 &= \mathrm{Sq}^4 x_5
\end{aligned}$$

and x_3, x_5 and x_9 are primitive.

The algebra structure of $\mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$ can be determined as an application of the Eilenberg-Moore spectral sequence. See [7].

Theorem 2.3.

$$\begin{aligned}
\mathrm{H}_*(\Omega G_2\ ;\mathbf{Z}/2\mathbf{Z}) &= \bigwedge(b_2)\otimes \mathbf{Z}/2\mathbf{Z}[b_4,b_{10}],\\
\mathrm{H}_*(\Omega F_4\ ;\mathbf{Z}/2\mathbf{Z}) &= \bigwedge(b_2)\otimes \mathbf{Z}/2\mathbf{Z}[b_4,b_{10},b_{14},b_{22}],\\
\mathrm{H}_*(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z}) &= \bigwedge(b_2)\otimes \mathbf{Z}/2\mathbf{Z}[b_4,b_8,b_{10},b_{14},b_{16},b_{22}],\\
\mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z}) &= \bigwedge(b_2,b_4,b_8)\otimes \mathbf{Z}/2\mathbf{Z}[b_{10},b_{14},b_{16},b_{18},b_{22},b_{26},b_{34}]
\end{aligned}$$

where b_i is a generator of degree i.

3. Adjoint action

Let $\mathrm{Ad} : G \times G \to G$ and $\mathrm{ad} : G \times \Omega G \to \Omega G$ be the adjoint action of a Lie group G defined by $\mathrm{Ad}(gh) = ghg^{-1}$ and $\mathrm{ad}(g,l)(t) = gl(t)g^{-1}$ where $g, h \in G$, $l \in \Omega G$ and $t \in [0,1]$. These induce the homomorphisms

$$\mathrm{Ad}_* : \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$$

and

$$\mathrm{ad}_* : \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}).$$

Put $y*y' = \mathrm{Ad}_*(y \otimes y')$ and $y*b = \mathrm{ad}_*(y \otimes b)$ where $y, y' \in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ and $b \in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$. Following are the dual statement of the result in [6].

Theorem 3.1. *For y, y', $y'' \in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ and b, $b' \in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$*

(i) $1 * y = y$, $1 * b = b$.
(ii) $y*1 = 0$, *if* $|y| > 0$, *whether* $1 \in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ *or* $1 \in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$.
(iii) $(yy') * b = y * (y' * b)$.
(iv) $y * (bb') = \sum (y' * b)(y'' * b')$ *where* $\Delta_* y = \sum y' \otimes y''$.
(v) $\sigma(y * b) = y * \sigma(b)$ *where* σ *is the homology suspension.*
(vi) $\mathrm{Sq}^n_*(y * b) = \sum_i (\mathrm{Sq}^i_* y) * (\mathrm{Sq}^{n-i}_* b)$.
$\mathrm{Sq}^n_*(y * y') = \sum_i (\mathrm{Sq}^i_* y) * (\mathrm{Sq}^{n-i}_* y')$.
(vii)

$$\begin{aligned} \Delta_*(y * b) &= (\Delta_* y) * (\Delta_* b) \\ &= \sum (y' * b') \otimes (y'' * b'') \end{aligned}$$

where $\Delta_* y = \sum y' \otimes y''$ *and* $\Delta_* b = \sum b' \otimes b''$. *Also*

$$\overline{\Delta}_*(y * b) = (\Delta_* y) * (\overline{\Delta}_* b).$$

(viii) *If* b *is primitive then* $y * b$ *is primitive.*

Also the result of [6] implies

Theorem 3.2. *We define a submodule A of $\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ as*

$$\begin{aligned} A &= \textstyle\bigwedge(y_6) && \textit{for } G = G_2, F_4, E_6 \\ A &= \textstyle\bigwedge(y_6, y_{10}, y_{18}) && \textit{for } G = E_7 \end{aligned}$$

where y_{2i} is the dual of x_i^2 with respect to the monomial basis. Then there exist a retraction $p : \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \to A$ and the following diagram commutes.

$$\begin{tikzcd}
\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \arrow[r, "ad_*"] \arrow[d, "p"] & \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \\
A \otimes \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \arrow[ur, "ad_*"] &
\end{tikzcd}$$

Proof. By Proposition 2.10 of [6] we have the folloing commutative diagram

$$\begin{tikzcd}
\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}^*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) & \mathrm{H}^*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \arrow[l, "\mathrm{ad}^*"'] \arrow[dl, "\mathrm{ad}^*"] \\
(\mathrm{T}_G^{2*} \cup 1) \otimes \mathrm{H}^*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z}) \arrow[u] &
\end{tikzcd}$$

where T_G^* is the set of all transgressive elements with respect to the principal fibration

$$G \to G/T \to BT.$$

Clearly

$$\begin{aligned}
\mathrm{T}_G^{2*} \cup 1 &= \bigwedge(x_3^2) \quad G = G_2, F_4, E_6, \\
\mathrm{T}_{E_7}^{2*} \cup 1 &= \bigwedge(x_3^2, x_5^2, x_9^2).
\end{aligned}$$

Using monomial basis of $\mathrm{H}^*(G;\mathbf{Z}/2\mathbf{Z})$ and T_G^{2*}, we can dualize the above result and regard $(\mathrm{T}_G^{2*})^* \cup 1$ as the submodule of $\mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$ and we obtain the statement.

Remark

1. By Theorem 3.1 (iv) and Theorem 3.2 we see that for $b \in \mathrm{H}_*(\Omega G\ ;\mathbf{Z}/2\mathbf{Z})$ and $i = 3, 5, 9$

$$\begin{aligned}
y_{2i} * b^2 &= (y_{2i} * b)b + (y_i * b)^2 + b(y_{2i} * b) \\
&= 0.
\end{aligned}$$

2. By theorem 3.1 and 3.2, when $G = G_2,\ F_4,\ E_6$ (resp. $G = E_7$), if $y_6 * b_j$ (resp. $y_6 * b_j,\ y_{10} * b_j$ and $y_{18} * b_j$) is determined for $b_j \in \mathrm{H}_*(G;\mathbf{Z}/2\mathbf{Z})$, then the map $\mathrm{H}_*(\mathrm{ad};\mathbf{Z}/2\mathbf{Z})$ is determined completely.

4. Adjoint action on ΩE_6

The next theorem is the main result for E_6 of this paper.

Theorem 4.1. *In Theorem 2.3 we can take b_i in $\mathrm{H}_*(\Omega E_6\,;\mathbf{Z}/2\mathbf{Z})$ so as to satisfy that*

(i)

$$\overline{\Delta}_*(b_i) = 0 \quad i \neq 4,\ 8,\ 16, \tag{1}$$

$$\overline{\Delta}_*(b_4) = b_2 \otimes b_2, \tag{2}$$

$$\overline{\Delta}_*(b_8) = b_2 \otimes b_2b_4 + b_4 \otimes b_4 + b_2b_4 \otimes b_2, \tag{3}$$

$$\begin{aligned}\overline{\Delta}_*(b_{16}) = {} & b_2 \otimes b_2b_4b_8 + b_4 \otimes b_4b_8 + b_2b_4 \otimes b_2b_8 + b_8 \otimes b_8 \\ & + b_2b_8 \otimes b_2b_4 + b_4b_8 \otimes b_4 + b_2b_4b_8 \otimes b_2 \\ & + b_2 \otimes b_2b_4^3 + b_2b_4^3 \otimes b_2 + b_4 \otimes b_4^3 + b_4^3 \otimes b_4.\end{aligned} \tag{4}$$

(ii)

$$\begin{aligned}&\mathrm{Sq}_*^2 b_4 = b_2,\ \mathrm{Sq}_*^2 b_8 = b_2b_4,\ \mathrm{Sq}_*^4 b_8 = b_4,\ \mathrm{Sq}_*^4 b_{16} = b_4b_8,\\ &\mathrm{Sq}_*^8 b_{16} = b_8,\ \mathrm{Sq}_*^2 b_{10} = b_4^2,\ \mathrm{Sq}_*^4 b_{10} = 0,\ \mathrm{Sq}_*^2 b_{14} = 0,\\ &\mathrm{Sq}_*^4 b_{14} = b_{10},\ \mathrm{Sq}_*^4 b_{22} = 0,\ \mathrm{Sq}_*^8 b_{22} = b_{14}.\end{aligned}$$

(iii)

$$\begin{aligned}&y_6 * b_2 = b_4^2,\ y_6 * b_4 = b_{10} + b_2b_4^2,\ y_6 * b_8 = b_{14} + b_{10}b_4 + b_4^3b_2,\\ &y_6 * b_{16} = b_{22} + b_{14}b_8 + b_{10}b_8b_4 + b_8b_4^3b_2 + b_{10}b_4^3 + b_4^5b_2,\\ &y_6 * b_{10} = b_4^4,\ y_6 * b_{14} = b_{10}^2,\ y_6 * b_{22} = b_{14}^2.\end{aligned}$$

Remark Theorem 4.1 states the whole informations of the Hopf algebra structure, the Steenrod algebra module structure and ad_* for $\mathrm{H}_*(\Omega E_6\,;\mathbf{Z}/2\mathbf{Z})$ except for $\mathrm{Sq}_*^2 b_{16}$ and $\mathrm{Sq}_*^2 b_{22}$. These are postponed until Theorem 5.2.

Proof of i). By Theorem 5.1 in [5] we see (1) and by Lemma 3.1 in [5] we can set

$$(b_2^*)^2 = b_4^*, \tag{5}$$

$$(b_2^*)^4 = b_8^*, \tag{6}$$

$$(b_2^*)^8 = b_{16}^*. \tag{7}$$

Here (5) implies (2). We set

$$\begin{aligned}&a_2 = b_2^*,\ a_8 = (b_4^2)^*,\ a_{16} = (b_4^4)^*,\\ &a_{10} = b_{10}^*,\ a_{14} = b_{14}^*\end{aligned}$$

where $(\ )^*$ means the dual with respect to the monomial basis of $\mathrm{H}_*(\Omega G\,;\mathbf{Z}/2\mathbf{Z})$. Then

$$\begin{aligned}\mathrm{H}_8(\Omega G\,;\mathbf{Z}/2\mathbf{Z}) &= \langle b_4^2,\ b_8\rangle,\\ \mathrm{H}^8(\Omega G\,;\mathbf{Z}/2\mathbf{Z}) &= \langle a_8,\ a_2^4\rangle.\end{aligned}$$

So we see

$$
\begin{aligned}
&(8) \qquad & a_8 &= (b_4^2)^* + p b_8^*, \\
&(9) \qquad & a_2^4 &= b_8^*
\end{aligned}
$$

where $p \in \mathbf{Z}/2\mathbf{Z}$. We can put $p = 0$ by re-defining b_8 by $b_8 + p b_4^2$. This implies (3). Also in $\mathrm{H}_{16}(\Omega G\ ; \mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}^{16}(\Omega G\ ; \mathbf{Z}/2\mathbf{Z})$ we know

$$
\begin{aligned}
\mathrm{H}_{16}(\Omega G\ ; \mathbf{Z}/2\mathbf{Z}) &= \langle b_{16},\ \ b_8^2,\ \ b_8 b_4^2,\ \ b_4^4,\ \ b_{14} b_2,\ \ b_{14} b_4 b_2 \rangle, \\
\mathrm{H}^{16}(\Omega G\ ; \mathbf{Z}/2\mathbf{Z}) &= \langle a_2^8,\ \ a_8^2,\ \ a_8 a_2^4,\ \ a_{16},\ \ a_{14} a_2,\ \ a_{14} a_2^3 \rangle,
\end{aligned}
$$

and we can see

$$
a_8^2 = (b_4^2)^* \cdot (b_4^2)^* = (b_4^2 \otimes b_4^2)^* \circ \Delta_*.
$$

This shows that $a_8^2 = (b_8^2)^* + q_1 b_{16}^*$ where $q_1 \in \mathbf{Z}/2\mathbf{Z}$. In the similar way we have

$$
(10) \quad \begin{aligned}
&a_8^2 = (b_8^2)^* + q_1 b_{16}^*,\ a_8 a_4^2 = (b_8 b_4^2)^* + q_2 b_{16}^*,\ a_{16} = (b_4^4)^* + q_3 b_{16}^*, \\
&a_{14} a_2 = (b_{14} b_2)^* + q_4 b_{16}^*,\ a_{10} a_2^3 = (b_{10} b_4 b_2)^* + q_5 b_{16}^*
\end{aligned}
$$

where $q_i \in \mathbf{Z}/2\mathbf{Z}$ for $1 \le i \le 5$. Again we re-define b_{16} by $b_{16} + q_1 b_8^2 + q_2 b_8 b_4^2 + q_3 b_4^4 + q_4 b_{14} b_2 + q_5 b_{10} b_4 b_2$ so that q_i becomes 0. Therefore by dualizing (7) and (10), the equations

$$
\begin{aligned}
a_2^4 a_8 &= a_2^2 (a_2^2 a_8) \\
&= b_4^* \cdot (b_4^* \cdot (b_4^2)^*) \\
&= b_4^* \cdot ((b_4 \otimes b_4^2)^* \circ \Delta_*) \\
&= (b_4 \otimes ((b_4^3) + (b_8 b_4))^* \circ \Delta_*
\end{aligned}
$$

and

$$
\begin{aligned}
a_2^4 a_8 &= a_2 (a_2^3 a_8) \\
&= (b_2 \otimes ((b_2 b_4^3) + (b_8 b_4 b_2))^* \circ \Delta_*
\end{aligned}
$$

deduce that

$$
\begin{aligned}
\overline{\Delta}_*(b_{16}) = \ & b_2 \otimes b_2 b_4 b_8 + b_4 \otimes b_4 b_8 + b_2 b_4 \otimes b_2 b_8 + b_8 \otimes b_8 \\
& + b_2 b_8 \otimes b_2 b_4 + b_4 b_8 \otimes b_4 + b_2 b_4 b_8 \otimes b_2 \\
& + b_2 \otimes b_2 b_4^3 + b_2 b_4^3 \otimes b_2 + b_4 \otimes b_4^3 + b_4^3 \otimes b_4.
\end{aligned}
$$

Proof of ii) and iii). By equations (5), (6), (7) and the above arguments we have easily

$$
\mathrm{Sq}_*^2 b_4 = b_2,\ \mathrm{Sq}_*^4 b_8 = b_4,\ \mathrm{Sq}_*^8 b_{16} = b_8.
$$

Also,

$$\begin{aligned}
\overline{\Delta}_*\mathrm{Sq}^2_* b_8 &= \mathrm{Sq}^2_*\overline{\Delta}_* b_8 \\
&= b_2\otimes b_4 + b_4\otimes b_2, \\
\overline{\Delta}_*\mathrm{Sq}^4_* b_{16} &= \mathrm{Sq}^4_*\overline{\Delta}_* b_{16} \\
&= b_2\otimes b_2b_8 + b_2b_8\otimes b_2 \\
&\quad +b_4\otimes b_8 + b_8\otimes b_4 \\
&\quad +b_2\otimes b_2b_4^2 + b_2b_4^2\otimes b_2 \\
&\quad +b_2b_4\otimes b_2b_4
\end{aligned}$$

and this implies that

$$\mathrm{Sq}^2_* b_8 = b_2b_4,\ \mathrm{Sq}^4_* b_{16} = b_4b_8 + b_4^3,$$

since there exists no primitive element in $\mathrm{H}_6(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_{12}(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z})$. Also we see

$$\begin{aligned}
\overline{\Delta}_*\mathrm{Sq}^2_* b_{16} &= \mathrm{Sq}^2_*\overline{\Delta}_* b_{16} \\
&= \overline{\Delta}_*(b_2b_4b_8 + b_2b_4^3)
\end{aligned}$$

and this implies

$$\mathrm{Sq}^2_* b_{16} = b_2b_4b_8 + b_2b_4^3 + (\text{primitive element}). \tag{11}$$

Next we consider $y_6 * b_i$. We start from the next lemma.

Lemma 4.2.

$$y_6 * b_2 = b_4^2.$$

Proof. We recall the exceptional Lie group G_2. By Theorem 2.1 and Theorem 2.2, we have

$$\mathrm{H}_*(G_2;\mathbf{Z}/2\mathbf{Z}) = \bigwedge(y_3, y_5, y_6)$$

where y_3, y_5 are the dual of x_3, x_5 and y_6 is the dual of x_3^2 with respect to the monomial basis of $\mathrm{H}^*(G_2;\mathbf{Z}/2\mathbf{Z})$. And by the inclusion $G_2 \to E_6$, y_i in $\mathrm{H}_*(G_2;\mathbf{Z}/2\mathbf{Z})$ and b_i in $\mathrm{H}_*(\Omega G_2\ ;\mathbf{Z}/2\mathbf{Z})$ corresponds to y_i in $\mathrm{H}_*(E_6;\mathbf{Z}/2\mathbf{Z})$ and b_i in $\mathrm{H}_*(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z})$. Therefore it is sufficient to prove that $y_6 * b_2 = b_4^2$ in the case of G_2.

There is an inclusion $SU(3)\xrightarrow{\kappa}G_2$ and

$$\mathrm{H}^*(SU(3);\mathbf{Z}/2\mathbf{Z}) = \bigwedge(x_3, x_5)$$

where $|x_i| = i$ and $x_5 = \mathrm{Sq}^2 x_3$. Also $\kappa^* x_3 = x_3$ and $\kappa^* x_5 = x_5$. We use the same notation for the elements which correspond by the inclusion.

First we observe the commutator map $\Gamma_0 : SU(3) \wedge SU(3) \to SU(3)$ and $\Gamma : G_2 \wedge G_2 \to G_2$. Here remember that there are the fibrations

$$\widetilde{SU}(3) \xrightarrow{i_0} SU(3) \xrightarrow{x_0} K(\mathbf{Z}, 3),$$

$$\widetilde{G_2} \xrightarrow{i} G_2 \xrightarrow{x} K(\mathbf{Z}, 3)$$

where x_0 and x represent the generator of $\mathrm{H}^3(SU(3); \mathbf{Z})$ and $\mathrm{H}^3(G_2; \mathbf{Z})$, and $\widetilde{SU}(3)$ and $\widetilde{G_2}$ are homotopy fibres of x_0 and x respectively.

Since $x_0 \circ \Gamma_0 \simeq *$ and $x \circ \Gamma \simeq *$, there are lifts $\widetilde{\Gamma}_0 : SU(3) \wedge SU(3) \to \widetilde{SU}(3)$ and $\widetilde{\Gamma} : G_2 \wedge G_2 \to \widetilde{G_2}$ such that $i_0 \circ \widetilde{\Gamma}_0 \simeq \Gamma_0$ and $i \circ \widetilde{\Gamma} \simeq \Gamma$. Also the following is known that

$$\mathrm{H}^*(\widetilde{SU}(3); \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}[x_8] \otimes \bigwedge(x_5', x_9)$$

$$\mathrm{H}^*(\widetilde{G_2}; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}[x_8] \otimes \bigwedge(x_9, x_{11})$$

where $|x_i| = i$ and $|x_5'| = 5$ and by inclusion $\widetilde{SU}(3) \xrightarrow{\widetilde{\kappa}} \widetilde{G_2}$ $\widetilde{\kappa}^* x_8 = x_8$ and $\widetilde{\kappa}^* x_9 = x_9$. (See [4].)

Next we introduce a subspace X of $SU(3) \wedge SU(3)$. We know that $SU(3) \simeq S^3 \cup e_5 \cup e_8$ and $S^3 \cup e_5 \simeq \Sigma \mathbf{CP}^2$ where e_i is a cell of degree i. We put

$$X = (S^3 \cup e_5) \wedge S^3 \simeq \Sigma \mathbf{CP}^2 \wedge S^3.$$

We can see easily that

$$\mathrm{H}^*(X; \mathbf{Z}/2\mathbf{Z}) = \langle \epsilon_6, \epsilon_8 \rangle$$

where $|\epsilon_i| = i$ and $\epsilon_8 = \mathrm{Sq}^2 \epsilon_6$.

We denote the 2-localization of $\widetilde{SU}(3)$ as $\widetilde{SU}(3)_{(2)}$ and the inclusion $\widetilde{SU}(3) \to \widetilde{SU}(3)_{(2)}$ as l_2. Then we have the following diagram:

$$\begin{tikzcd}
 & & \widetilde{SU}(3) \arrow[r, "l_2"] \arrow[d, "i_0"] & \widetilde{SU}(3)_{(2)} \\
X = \Sigma \mathbf{CP}^2 \wedge S^3 \arrow[r, hook] & SU(3) \wedge SU(3) \arrow[ur, "\widetilde{\Gamma}_0"] \arrow[r, "\Gamma_0"] & SU(3) \arrow[d] & \\
 & & K(\mathbf{Z}, 3) &
\end{tikzcd}$$

Let f be the map $f : X \to \widetilde{SU}(3)_{(2)}$ defined by $f = l_2 \circ \widetilde{\Gamma}_0|_X$.

We can see easily $\pi_5(\widetilde{SU}(3)_{(2)}) = \mathbf{Z}/2\mathbf{Z}$. Let $\alpha : {S^5}_{(2)} \to \widetilde{SU}(3)_{(2)}$ be the 2-localization of its generator. Then $\alpha_* : \mathrm{H}_*({S^5}_{(2)}; \mathbf{Z}) \to \mathrm{H}_*(\widetilde{SU}(3)_{(2)}; \mathbf{Z})$ is isomorphic for $* \leq 6$ and epic for $* = 7$. Thus by Whitehead's theorem

$$\alpha_* : \pi_6({S^5}_{(2)}) \xrightarrow{\cong} \pi_6(\widetilde{SU}(3)_{(2)}) \tag{12}$$

is isomorphic.

Here we refer to R.Bott's result that

$$\Gamma_0|_{S^3 \wedge S^3} \in \pi_6(SU(3)) \cong \mathbf{Z}/6\mathbf{Z}$$

is a generator. (See [3].) This implies $f|_{S^3 \wedge S^3} \in \pi_6(\widetilde{SU}(3)_{(2)}) \cong \mathbf{Z}/2\mathbf{Z}$ is the generator. Thus (12) implies that there exists a map

$$g : S^6 \to {S^5}_{(2)}$$

and g represents the generator of $\pi_6({S^5}_{(2)}) \cong \mathbf{Z}/2\mathbf{Z}$ and the following diagram commutes upto homotopy.

$$\begin{tikzcd} X \arrow[r, "f"] & \widetilde{SU}(3)_{(2)} \\ S^6 \arrow[u, "\iota"] \arrow[r, "g"] & {S^5}_{(2)} \arrow[u, "\alpha"'] \end{tikzcd}$$

Lemma 4.3.

$$f^*(x_8) = \epsilon_8.$$

Proof. We assume $f^*(x_8) = 0$. Let C_f and C_g be the mapping cone of f and g respectively. Consider the commutative diagram below.

$$\begin{array}{ccccccccc} X & \xrightarrow{f} & \widetilde{SU}(3)_{(2)} & \xrightarrow{k} & C_f & \xrightarrow{j} & \Sigma X & \to \cdots \\ \uparrow \iota & & \uparrow \alpha & & \uparrow \iota' & & \uparrow \Sigma\iota & \\ S^6 & \xrightarrow{g} & {S^5}_{(2)} & \xrightarrow{k'} & C_g & \xrightarrow{j'} & \Sigma S^6 & \to \cdots \end{array}$$

Then we can see

$$\mathrm{H}^*(C_f; \mathbf{Z}/2\mathbf{Z}) = \langle \bar{x}_5, \bar{x}_8, \bar{x}_9, \bar{\epsilon}_7, \bar{\epsilon}_9 \rangle \text{ for } * < 10,\ |\bar{x}_i| = i,\ |\bar{\epsilon}_i| = i$$

where $k^*(\bar{x}_i) = x_i$ and $j^*(\Sigma\epsilon_i) = \bar{\epsilon}_{i+1}$. Also we can show easily

$$\mathrm{H}^*(C_g; \mathbf{Z}/2\mathbf{Z}) = \langle \bar{c}_5, \bar{c}_7 \rangle,\ |\bar{c}_i| = i$$

and $k'^*(\bar{c}_5) = c_5$ and $j'^*(\Sigma c_6) = \bar{c}_7$ where c_i is the generator of $\mathrm{H}^i(S^i; \mathbf{Z}/2\mathbf{Z})$. Then we have the equations

$$\iota^*(\epsilon_6) = c_6,\ \alpha^*(x_5) = c_5,$$

$$\iota'^*(\bar{x}_5) = \bar{c}_5,\ \iota'^*(\bar{\epsilon}_7) = c_7.$$

Recall that $[g]$ is the generator of $\pi_6({S^5}_{(2)}) \cong \pi_6(\widetilde{SU}(3)_{(2)}) \cong \mathbf{Z}/2\mathbf{Z}$. This implies that the 2-localization of g, $g_{(2)} : {S^6}_{(2)} \to {S^5}_{(2)}$ is homotopic to $\Sigma^3\gamma_{(2)}$ where γ is the Hopf map $\gamma : S^3 \to S^2$. Thus $C_{g_{(2)}} \simeq \Sigma^3 C_{\gamma_{(2)}} \simeq \Sigma^3 {\mathbf{CP}^2}_{(2)}$ and we have

$$\mathrm{Sq}^2 \bar{c}_5 = \bar{c}_7 \quad \text{ in } \mathrm{H}^*(C_g; \mathbf{Z}/2\mathbf{Z})\ .$$

Therefore $\mathrm{Sq}^2\bar{x}_5 = \bar{\epsilon}_7$, since, if it were not, $\bar{c}_7 = \mathrm{Sq}^2_*\bar{c}_5 = \mathrm{Sq}^2_*\iota'^*\bar{x}_5 = \iota'^*(\mathrm{Sq}^2_*\bar{x}_5) = 0$. We easily see $\mathrm{Sq}^2_*\bar{\epsilon}_7 = \bar{\epsilon}_9$ also.

On the other hand, by the Adem relation, we obtain

$$\mathrm{Sq}^2\mathrm{Sq}^2\bar{x}_5 = \mathrm{Sq}^3\mathrm{Sq}^1\bar{x}_5 = 0.$$

These contradict each other. Thus $f^*(x_8) = \epsilon_8$.

Q.E.D.(Lemma 4.3)

Since Lemma 4.3 implies ${\widetilde{\Gamma}_0}^*(x_8) \neq 0$, the only one possibility is

$${\widetilde{\Gamma}_0}^*(x_8) = x_3 \otimes x_5 + x_5 \otimes x_3.$$

Then by the naturality of the commutator, we have

$$\widetilde{\Gamma}^*(x_8) = x_3 \otimes x_5 + x_5 \otimes x_3$$

and

$$\begin{aligned} \widetilde{\Gamma}^*(x_9) &= \widetilde{\Gamma}(\mathrm{Sq}^1 x_8) \\ &= \mathrm{Sq}^1(x_3 \otimes x_5 + x_5 \otimes x_3) \\ &= x_3 \otimes x_3^2 + x_3^2 \otimes x_3. \end{aligned}$$

By dualizing this, we have

$$\widetilde{\Gamma}_*(y_6 \otimes y_3) = y_9 \tag{13}$$

where $y_9 \in \mathrm{H}_*(\widetilde{G_2}; \mathbf{Z}/2\mathbf{Z})$ is the dual element of $x_9 \in \mathrm{H}^*(\widetilde{G_2}; \mathbf{Z}/2\mathbf{Z})$ with respect to the monomial basis.

Now we consider the case of $\Omega\widetilde{G_2}$. We have the fibration

$$\Omega\widetilde{G_2} \to \Omega G_2 \to K(\mathbf{Z}, 2)$$

and the commutator map $\Gamma' : G_2 \wedge \Omega G_2 \to \Omega G_2$ lifts to the map $\widetilde{\Gamma}' : G_2 \wedge \Omega G_2 \to \Omega\widetilde{G_2}$. Here we can set

$$\widetilde{\Gamma}'(g, l)(t) = \widetilde{\Gamma}(g, l(t))$$

for $g \in G$, $l \in \Omega G$ and $t \in [0, 1]$. Thus we have the following commutative diagram in which the coefficient ring $\mathbf{Z}/2\mathbf{Z}$ is abbreviated.

$$\begin{tikzcd}
& & \mathrm{H}_{*+1}(\widetilde{G_2}) \arrow[d, "i_*"] & \\
\mathrm{H}_*(G_2)\otimes \mathrm{H}_{*+1}(G_2) \arrow[urr, "\widetilde{\Gamma}_*"] \arrow[rr, "\Gamma_*"'] & & \mathrm{H}_{*+1}(G_2) & \\
& & & \mathrm{H}_*(\Omega\widetilde{G_2}\ ) \arrow[uul, "\sigma"'] \arrow[d, "\Omega i\ _*"] \\
& \mathrm{H}_*(G_2)\otimes \mathrm{H}_*(\Omega G_2\ ) \arrow[uul, "\mathrm{Id}\otimes\sigma"] \arrow[urr, "\widetilde{\Gamma}'_*"] \arrow[rr, "\Gamma'_*"'] & & \mathrm{H}_*(\Omega G_2\ ) \arrow[uul, "\sigma"']
\end{tikzcd}$$

Also, we know that

$$\mathrm{H}_*(\Omega\widetilde{G_2}\ ; \mathbf{Z}/2\mathbf{Z}) = \bigwedge(b_7') \otimes \mathbf{Z}/2\mathbf{Z}[b_8', b_{10}]$$

and $\Omega i\ _*(b_8') = b_4^2$, $\Omega i\ _*(b_{10}) = b_{10}$ and $\sigma(b_8') = y_9$. This can be seen by the Serre spectral sequence of the fibration $S^1 \to \Omega\widetilde{G_2}\ \to \Omega G$.

Thus (13) implies that

$$y_9 = \widetilde{\Gamma}_*(y_6 \otimes \sigma(b_2)) = \sigma\widetilde{\Gamma}'_*(y_6 \otimes b_2).$$

Then $\widetilde{\Gamma}'_*(y_6 \otimes b_2) \neq 0$, that is, $\widetilde{\Gamma}'_*(y_6 \otimes b_2) = b_8'$. Therefore

$$\begin{aligned} \Gamma'_*(y_6 \otimes b_2) &= \Omega i\ _* \circ \widetilde{\Gamma}'_*(y_6 \otimes b_2) \\ &= \Omega i\ _* b_8' = b_4^2. \end{aligned}$$

Since the following diagram commutes,

$$\Gamma'_*(y_6 \otimes b_2) = (y_6 * 1)\cdot b_2 + (y_6 * b_2)\cdot 1 = y_6 * b_2.$$

$$\begin{tikzcd}
G_2 \times \Omega G_2 \arrow[r, "\Gamma\text{'}"] \arrow[d, "1\times\Delta"'] & \Omega G_2 \\
G_2 \times \Omega G_2\ \times \Omega G_2 \arrow[r, "\mathrm{ad}\times 1"] & \Omega G_2\ \times \Omega G_2 \arrow[u, "\lambda"']
\end{tikzcd}$$

Thus we finally obtain

$$y_6 * b_2 = b_4^2.$$

Q.E.D.(Lemma 4.2)

We remark that $y_6 * b_i$ can be determined upto primitive elements, if all $y_6 * b'$ and $y_6 * b''$ are determined where $\overline{\Delta}_* b_i = \sum b' \otimes b''$. Since

$$\begin{aligned} \overline{\Delta}_* y_6 * b_i &= (y_6 \otimes 1 + y_3 \otimes y_3 + 1 \otimes y_6) * \overline{\Delta}_* b_i \\ &= \sum (y_6 * b') \otimes b'' + b' \otimes (y_6 * b''). \end{aligned}$$

For example, since $\overline{\Delta}_* y_6 * b_4 = (y_6 * b_2) \otimes b_2 + b_2 \otimes (y_6 * b_2) = \overline{\Delta}_*(b_2 b_4^2)$,

$$y_6 * b_4 = \rho_{(6,4)} b_{10} + b_2 b_4^2$$

where $\rho_{(6,4)} \in \mathbf{Z}/2\mathbf{Z}$. Then we have

(14)

$$\begin{aligned}
&y_6 * b_4 = \rho_{(6,4)} b_{10} + b_2 b_4^2,\\
&y_6 * b_8 = \rho_{(6,8)} b_{14} + b_4(y_6 * b_4) + b_2 b_4^3,\\
&y_6 * b_{16} = \rho_{(6,16)} b_{22} + b_8(y_6 * b_8) + (b_4 b_8 + b_4^3)(y_6 * b_4) + b_2 b_4^3 b_8 + b_2 b_4^5
\end{aligned}$$

where $\rho_{(6,i)} \in \mathbf{Z}/2\mathbf{Z}$.

On the other hand, we have

$$\begin{aligned}
\mathrm{Sq}_*^2(y_6 * b_4) &= y_6 * (\mathrm{Sq}_*^2 b_4) = y_6 * b_2,\\
\mathrm{Sq}_*^4(y_6 * b_8) &= y_6 * (\mathrm{Sq}_*^4 b_8) = y_6 * b_4,\\
\mathrm{Sq}_*^8(y_6 * b_{16}) &= y_6 * (\mathrm{Sq}_*^8 b_{16}) = y_6 * b_8.
\end{aligned} \tag{15}$$

Since Steenrod operators map primitive elements into primitive elements and decomposable elements into decomposable elements, by (14), (15) and Lemma 4.2 we obtain that

$$\rho_{(6,4)} \mathrm{Sq}_*^2 b_{10} = b_4^2,\ \rho_{(6,8)} \mathrm{Sq}_*^4 b_{14} = \rho_{(6,4)} b_{10},\ \rho_{(6,16)} \mathrm{Sq}_*^8 b_{22} = \rho_{(6,8)} b_{14}$$

and this implies that

$$\rho_{(6,4)} = \rho_{(6,8)} = \rho_{(6,16)} = 1,$$

$$\mathrm{Sq}_*^2 b_{10} = b_4^2,\ \mathrm{Sq}_*^4 b_{14} = b_{10},\ \mathrm{Sq}_*^8 b_{22} = b_{14}. \tag{16}$$

Therefore by (14) we have that

$$\begin{aligned}
&y_6 * b_4 = b_{10} + b_2 b_4^2,\ y_6 * b_8 = b_{14} + b_{10} b_4 + b_4^3 b_2,\\
&y_6 * b_{16} = b_{22} + b_{14} b_8 + b_{10} b_8 b_4 + b_8 b_4^3 b_2 + b_{10} b_4^3 + b_4^5 b_2.
\end{aligned}$$

Since b_{14} and b_{22} are primitive, we have the equations

$$\begin{aligned}
y_6 * b_{14} &= \rho_{(6,14)} b_{10}^2,\\
y_6 * b_{22} &= \rho_{(6,22)} b_{14}^2
\end{aligned} \tag{17}$$

where $\rho_{(6,i)} \in \mathbf{Z}/2\mathbf{Z}$. On the other hand by (16) we have

$$\begin{aligned}
\mathrm{Sq}_*^4(y_6 * b_{14}) &= y_6 * \mathrm{Sq}_*^4 b_{14} = y_6 * b_{10},\\
\mathrm{Sq}_*^8(y_6 * b_{22}) &= y_6 * \mathrm{Sq}_*^8 b_{22} = y_6 * b_{14}.
\end{aligned} \tag{18}$$

Since

$$\begin{aligned}
0 &= (y_6^2) * b_4\\
&= y_6 * (y_6 * b_4)\\
&= y_6 * b_{10} + y_6 * (b_2 b_4^2),
\end{aligned}$$

we obtain

$$y_6 * b_{10} = b_4^4.$$

Therefore (17) and (18) implies that

$$\rho_{(6,14)} = \rho_{(6,22)} = 1.$$

Since there is no primitive elements in $\mathrm{H}_6(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_{18}(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z})$ and since b_{10} and b_{22} are primitive, we have

$$\mathrm{Sq}^4_* b_{10} = 0,\ \ \mathrm{Sq}^4_* b_{22} = 0.$$

Thus we get the all formulas in Theorem 4.1.

Q.E.D.

By Theorem 4.1 we can deduce the following theorem about G_2 and F_4.

Theorem 4.4. 1. *In* $\mathrm{H}_*(\Omega G_2\ ;\mathbf{Z}/2\mathbf{Z})$

$$\begin{aligned} y_6 * b_2 &= b_4^2, \\ y_6 * b_4 &= b_{10} + b_2 b_4^2, \\ y_6 * b_{10} &= b_4^4. \end{aligned}$$

2. *In* $\mathrm{H}_*(\Omega F_4\ ;\mathbf{Z}/2\mathbf{Z})$

$$\begin{aligned} y_6 * b_2 &= b_4^2, \\ y_6 * b_4 &= b_{10} + b_2 b_4^2, \\ y_6 * b_{10} &= b_4^4, \\ y_6 * b_{14} &= b_{10}^2, \\ y_6 * b_{22} &= b_{14}^2. \end{aligned}$$

Proof. By the naturality of the adjoint action we have the following commutative diagram.

$$\begin{array}{ccc} G_2 \times \Omega G_2 & \xrightarrow{\mathrm{ad}} & \Omega G_2 \\ \downarrow & & \downarrow \\ F_4 \times \Omega F_4 & \xrightarrow{\mathrm{ad}} & \Omega F_4 \\ \downarrow & & \downarrow \\ E_6 \times \Omega E_6 & \xrightarrow{\mathrm{ad}} & \Omega E_6 \end{array}$$

Here $\mathrm{H}_*(\Omega G_2\ ;\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_*(\Omega F_4\ ;\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{H}_*(\Omega F_4\ ;\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_*(\Omega E_6\ ;\mathbf{Z}/2\mathbf{Z})$ are monic. Then Theorem 4.1 implies the statements.

5. Adjoint action on ΩE_7

For the Hopf algebra structure of $\mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z})$ we refer to the following result of [5].

Theorem 5.1 (A.Kono & K.Kozima). *In Theorem 2.3 we can choose b_i in $\mathrm{H}_*(\Omega E_7\ ;\mathbf{Z}/2\mathbf{Z})$ so as to satisfy that*

$$
\begin{array}{lrcl}
(19) & \overline{\Delta}_*(b_i) & = & 0 \quad \textit{for } i \neq 4,\ 8,\ 16, \\
(20) & \overline{\Delta}_*(b_4) & = & b_2 \otimes b_2, \\
(21) & \overline{\Delta}_*(b_8) & = & b_2 \otimes b_2b_4 + b_4 \otimes b_4 + b_2b_4 \otimes b_2, \\
 & \overline{\Delta}_*(b_{16}) & = & b_2 \otimes b_2b_4b_8 + b_4 \otimes b_4b_8 + b_2b_4 \otimes b_2b_8 + b_8 \otimes b_8 \\
(22) & & & +b_2b_8 \otimes b_2b_4 + b_4b_8 \otimes b_4 + b_2b_4b_8 \otimes b_2.
\end{array}
$$

Proof. For (19) see Theorem 5.1 in [5]. Then (20), (21) and (22) follows from Theorem 4.1.

Now we observe the induced homomorphism on homology by the adjoint action of E_7 on ΩE_7 .

Theorem 5.2. *In Theorem 5.1 b_i satisfies the following tables*

b_i	$y_6 * b_i$	$y_{10} * b_i$	$y_{18} * b_i$
b_2	0	0	b_{10}^2
b_4	b_{10}	b_{14}	$b_{22} + b_2b_{10}^2$
b_8	$b_{14} + b_4b_{10}$	$b_{18} + b_4b_{14}$	$b_{26} + b_4b_{22} + b_2b_4b_{10}^2$
b_{10}	0	b_{10}^2	b_{14}^2
b_{14}	b_{10}^2	0	b_{16}^2
b_{16}	$b_{22} + b_8b_{14} + b_4b_8b_{10}$	$b_{26} + b_8b_{18} + b_4b_8b_{14}$	$b_{34} + b_8b_{26} + b_4b_8b_{22} + b_2b_4b_8b_{10}^2$
b_{18}	0	b_{14}^2	b_{18}^2
b_{22}	b_{14}^2	b_{16}^2	b_{10}^4
b_{26}	b_{16}^2	b_{18}^2	b_{22}^2
b_{34}	b_{10}^4	b_{22}^2	b_{26}^2

b_i	$\mathrm{Sq}_*^2 b_i$	$\mathrm{Sq}_*^4 b_i$	$\mathrm{Sq}_*^8 b_i$	$\mathrm{Sq}_*^{16} b_i$
b_4	b_2	—	—	—
b_8	b_2b_4	b_4	—	—
b_{10}	b_4^2	0	—	—
b_{14}	0	b_{10}	—	—
b_{16}	$b_{14} + b_2b_4b_8$	b_4b_8	b_8	—
b_{18}	0	0	b_{10}	—
b_{22}	b_{10}^2	0	b_{14}	—
b_{26}	0	b_{22}	b_{18}	—
b_{34}	b_{16}^2	0	0	b_{18}

Proof. By considering the inclusion $E_6 \to E_7$, the result of Theorem 4.1 turns into

$$\begin{aligned}
&y_6 * b_2 = 0,\ y_6 * b_4 = b_{10},\ y_6 * b_4 = b_{10},\ y_6 * b_8 = b_{14} + b_4 b_{10},\ y_6 * b_{10} = 0,\\
&y_6 * b_{16} = b_{22} + b_8 b_{14} + b_4 b_8 b_{10},\ y_6 * b_{14} = b_{10}^2,\ y_6 * b_{22} = b_{14}^2,\\
&\mathrm{Sq}_*^2 b_4 = b_2,\ \mathrm{Sq}_*^2 b_8 = b_2 b_4,\ \mathrm{Sq}_*^4 b_8 = b_4,\ \mathrm{Sq}_*^4 b_{16} = b_4 b_8,\\
&\mathrm{Sq}_*^8 b_{16} = b_8,\ \mathrm{Sq}_*^2 b_{10} = 0,\ \mathrm{Sq}_*^4 b_{10} = 0,\ \mathrm{Sq}_*^2 b_{14} = 0,\\
&\mathrm{Sq}_*^4 b_{14} = b_{10},\ \mathrm{Sq}_*^4 b_{22} = 0,\ \mathrm{Sq}_*^8 b_{22} = b_{14}.
\end{aligned}$$

If b_i is primitive, $y_6 * b_i$, $y_{10} * b_i$, $y_{18} * b_i$ and $\mathrm{Sq}_*^j b_i$ are primitive . Thus

$$y_i * b_j = 0 \text{ for } (i,j) = (6,18),\ (10,2),\ (10,14)$$

and

$$\mathrm{Sq}_*^j * b_i = 0 \text{ for } (i,j) = (18,2),\ (26,2),\ (34,4)$$

since there is no primitive elements of degrees which these elements have in $\mathrm{H}_i(\Omega E_7\ ; \mathbf{Z}/2\mathbf{Z})$.

As stated in the proof of Theorem 4.1, $y_6 * b_i$ can be determined modulo primitive elements, if all $y_6 * b'$ and $y_6 * b''$ are known where $\overline{\Delta}_* b_i = \sum b' \otimes b''$. This is true for the case of $y_{10} * b_i$ and $y_{18} * b_i$. Thus we can put as follows:

$$y_{10} * b_4 = \rho_{(10,4)} b_{14}, \tag{23}$$

$$y_{10} * b_8 = \rho_{(10,8)} b_{18} + b_4 (y_{10} * b_4), \tag{24}$$

$$y_{10} * b_{16} = \rho_{(10,16)} b_{26} + (\text{decomposable elements}) \tag{25}$$

where $\rho_{(10,i)} \in \mathbf{Z}/2\mathbf{Z}$. By applying Sq_*^4 for (23), obtain

$$\rho_{(10,4)} \mathrm{Sq}_*^4 b_{14} = \mathrm{Sq}_*^4 (y_{10} * b_4) = y_6 * b_4 = y_{10}$$

and this implies $\rho_{(10,4)} = 1$. Also by applying Sq_*^8 for (24) and Sq_*^4 for (25), we obtain the following equations in the similarway:

$$\begin{aligned}
\rho_{(10,8)} \mathrm{Sq}_*^8 b_{18} &= \mathrm{Sq}_*^8 (y_{10} * b_8 + b_4 b_{14})\\
&= y_6 * b_4\\
&= y_{10}, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
\rho_{(10,16)} \mathrm{Sq}_*^4 b_{26} &= \mathrm{Sq}_*^4 (y_{10} * b_{16})\\
&= y_6 * b_{16} + y_{10} * (b_4 b_8)\\
&= b_{22} \qquad \text{mod decomposable elements.} \qquad (27)
\end{aligned}$$

Then (26) and (27) implies $\rho_{(10,8)} = 1$ and $\rho_{(10,16)} = 1$ and $\mathrm{Sq}_*^8 b_{18} = b_{10}$. Also, since $\mathrm{Sq}_*^4 b_{26}$ is primitive and no decomposable element in

$\mathrm{H}_{22}(\Omega E_7\,;\mathbf{Z}/2\mathbf{Z})$ is primitive, (27) tells that $\mathrm{Sq}_*^4 b_{26} = b_{22}$. Therefore we obtain

$$
\begin{aligned}
&(28) \qquad & y_{10} * b_4 &= b_{14}, \\
&(29) \qquad & y_{10} * b_8 &= b_{18} + b_4 b_{14}, \\
&(30) \qquad & y_{10} * b_{16} &= b_{26} + b_8 b_{18} + b_4 b_8 b_{14}.
\end{aligned}
$$

By applying Sq_*^2 and Sq_*^4 to (29) and Sq_*^8 to (30), we have

$$y_{10} * (b_2 b_4) = \mathrm{Sq}_*^2 b_{18} + b_2 b_{14},$$

$$y_6 * b_8 + y_{10} * b_4 = \mathrm{Sq}_*^4 b_{18} + b_4 b_{10},$$

$$y_6 * (b_4 b_8) + y_{10} * b_8 = \mathrm{Sq}_*^8 b_{26} + b_8 b_{10}.$$

So we obtain that

$$\mathrm{Sq}_*^2 b_{18} = 0,\ \mathrm{Sq}_*^4 b_{18} = 0,\ \mathrm{Sq}_*^8 b_{26} = b_{18}.$$

Also, $y_{10} * b_{10}$ can be computed as

$$
\begin{aligned}
y_{10} * b_{10} &= y_{10} * (y_6 * b_4) \\
&= y_6 * (y_{10} * b_4) \\
&= y_6 * b_{14} = b_{10}^2.
\end{aligned}
$$

Next we observe $y_{10} * b_{18}$, $y_{10} * b_{22}$ and $y_{10} * b_{26}$. Since $y_{10} * b_{18}$ is primitive, we can put

$$y_{10} * b_{18} = \rho_{(10,18)} b_{14}^2, \tag{31}$$

$$y_{10} * b_{22} = \rho_{(10,22)} b_{16}^2, \tag{32}$$

$$y_{10} * b_{26} = \rho_{(10,26)} b_{18}^2 \tag{33}$$

where $\rho_{(10,i)} \in \mathbf{Z}/2\mathbf{Z}$. By applying Sq_*^8 for (31), Sq_*^4 for (32) and Sq_*^{16} for (33), we obtain that

$$\rho_{(10,18)} \mathrm{Sq}_*^8 b_{14}^2 = \mathrm{Sq}_*^8 y_{10} * b_{18} = y_{10} * b_{10} = b_{10}^2,$$

$$\rho_{(10,22)} \mathrm{Sq}_*^4 b_{16}^2 = \mathrm{Sq}_*^4 y_{10} * b_{22} = y_6 * b_{22} = b_{14}^2,$$

$$\rho_{(10,26)} \mathrm{Sq}_*^{16} b_{18}^2 = \mathrm{Sq}_*^{16} y_{10} * b_{26} = y_6 * b_{14} = b_{10}^2.$$

Therefore we have $\rho_{(10,18)} = \rho_{(10,22)} = \rho_{(10,26)} = 1$ and

$$\mathrm{Sq}_*^4(b_{16}^2) = b_{14}^2. \tag{34}$$

Remember that by (11) in the proof of Theorem 4.1 we have

$$\mathrm{Sq}_*^2 b_{16} = k b_{14} + b_2 b_4 b_8 + b_2 b_4^3 \text{ in } \mathrm{H}_*(\Omega E_6\,;\mathbf{Z}/2\mathbf{Z})$$

where $k \in \mathbf{Z}/2\mathbf{Z}$ and then

$$\mathrm{Sq}_*^2 b_{16} = kb_{14} + b_2 b_4 b_8 \text{ in } \mathrm{H}_*(\Omega E_7\,;\mathbf{Z}/2\mathbf{Z}).$$

Then one can easily show $k = 1$ from (34). Hence

$$\begin{array}{ll} \mathrm{Sq}_*^2 b_{16} = b_{14} + b_2 b_4 b_8 + b_2 b_4^3 & \text{in } \mathrm{H}_*(\Omega E_6\,;\mathbf{Z}/2\mathbf{Z}), \\ \mathrm{Sq}_*^2 b_{16} = b_{14} + b_2 b_4 b_8 & \text{in } \mathrm{H}_*(\Omega E_7\,;\mathbf{Z}/2\mathbf{Z}). \end{array}$$

Moreover we have that in $\mathrm{H}_*(\Omega E_6\,;\mathbf{Z}/2\mathbf{Z})$

$$\begin{aligned} \mathrm{Sq}_*^2(y_6 * b_{16}) &= y_6 * (b_{14} + b_2 b_4 b_8 + b_2 b_4^3) \\ &= b_{10}^2 + b_2 b_4 b_{14} + b_2 b_8 b_{10} + b_4^3 b_8 + b_4^5, \end{aligned}$$

while

$$\begin{aligned} \mathrm{Sq}_*^2(y_6 * b_{16}) &= \mathrm{Sq}_*^2(b_{22} + b_8 b_{14} + b_4 b_8 b_{10} + b_4^3 b_{10} + b_2 b_4^5) \\ &= \mathrm{Sq}_*^2 b_{22} + b_2 b_4 b_{14} + b_2 b_8 b_{10} + b_4^3 b_8 + b_4^5. \end{aligned}$$

Therefore it follows that

$$\mathrm{Sq}_*^2 b_{22} = b_{10}^2 \text{ in } \mathrm{H}_*(\Omega E_6\,;\mathbf{Z}/2\mathbf{Z}) \text{ and } \mathrm{H}_*(\Omega E_7\,;\mathbf{Z}/2\mathbf{Z}).$$

Next we consider $y_{18} * b_2$, $y_{18} * b_4$ and $y_{18} * b_8$. We can put

(35) $$y_{18} * b_2 = \rho_{(18,2)} b_{10}^2,$$

(36) $$y_{18} * b_4 = \rho_{(18,4)} b_{22} + (\text{decomposable elements}),$$

(37) $$y_{18} * b_8 = \rho_{(18,8)} b_{26} + (\text{decomposable elements}),$$

(38) $$y_{18} * b_{16} = \rho_{(18,16)} b_{34} + (\text{decomposable elements}).$$

By applying Sq_*^8 to (37), we have

$$\begin{aligned} \rho_{(18,8)} \mathrm{Sq}_*^8 b_{26} &\equiv \mathrm{Sq}_*^8(y_{18} * b_8) \\ &\equiv y_{10} * b_8 \\ &\equiv b_{18} \bmod \text{decomposable elements}. \end{aligned}$$

Thus $\rho_{(18,8)} = 1$ and also we see

$$\begin{aligned} y_{18} * b_4 &= \mathrm{Sq}_*^4(y_{18} * b_8) \\ &\equiv \mathrm{Sq}_*^4 b_{26} \\ &\equiv b_{22} \bmod \text{decomposable elements}. \end{aligned}$$

This means $\rho_{(18,4)} = 1$. Moreover we know that

$$\overline{\Delta}_*(y_{18} * b_4) = b_2 \otimes (y_{18} * b_2) + (y_{18} * b_2) \otimes b_2,$$

that is, $y_{18} * b_4 = b_{22} + b_2(y_{18} * b_2)$. Therefore

$$\begin{aligned} y_{18} * b_2 &= \mathrm{Sq}_*^2(y_{18} * b_4) \\ &= \mathrm{Sq}_*^2(b_{22} + \rho_{(18,2)} b_2 b_{10}^2) \\ &= b_{10}^2 \end{aligned}$$

and $\rho_{(18,2)} = 1$. Also operating Sq_*^{16} to (38), we see

$$\begin{aligned} y_{10} * b_8 &= \mathrm{Sq}_*^{16}(y_{18} * b_{16}) \\ &= \rho_{(18,16)}\mathrm{Sq}_*^{16}b_{34} + (\text{decomposable elements}). \end{aligned}$$

Then, by (29), we deduce $\rho_{(18,16)} = 1$ and $\mathrm{Sq}_*^{16}b_{34} = b_{18}$.

Now we can compute $y_{18} * b_2, y_{18} * b_4, y_{18} * b_8$ and $y_{18} * b_{16}$, using

$$y_{18} * b_4 = \rho_{(18,4)}b_{22} + b_2(y_{18} * b_2)$$

and by the similar manner. Hence we have

$$\begin{aligned} &(39) & y_{18} * b_2 &= b_{10}^2, \\ &(40) & y_{18} * b_4 &= b_{22} + b_2 b_{10}^2, \\ &(41) & y_{18} * b_8 &= b_{26} + b_4 b_{22} + b_2 b_4 b_{10}^2, \\ &(42) & y_{18} * b_{16} &= b_{34} + b_8 b_{26} + b_4 b_8 b_{22} + b_2 b_4 b_8 b_{10}^2. \end{aligned}$$

Next we observe $y_{18} * b_{10}$, $y_{18} * b_{14}$, $y_{18} * b_{18}$ and $y_{18} * b_{26}$. We can put

$$\begin{aligned} &(43) & y_{18} * b_{10} &= \rho_{(18,10)}b_{14}^2, \\ &(44) & y_{18} * b_{14} &= \rho_{(18,14)}b_{16}^2, \\ &(45) & y_{18} * b_{18} &= \rho_{(18,18)}b_{18}^2, \\ &(46) & y_{18} * b_{26} &= \rho_{(18,26)}b_{22}^2, \end{aligned}$$

by primitivity. We can easily show $\rho_{(18,10)} = \rho_{(18,14)} = \rho_{(18,18)} = \rho_{(18,26)} = 1$ by applying Sq_*^8 to (43), Sq_*^4 to (44), Sq_*^{16} to (45) and Sq_*^{16} to (46). Also by applying Sq_*^4 to (46), we have

$$y_{18} * b_{22} = \mathrm{Sq}_*^4(y_{18} * b_{26}) = \mathrm{Sq}_*^4 b_{22}^2 = b_{10}^4.$$

Now the rest we have to do is to determine $y_6 * b_{34}$, $y_{10} * b_{34}, y_{18} * b_{34}$ and to determine $\mathrm{Sq}_*^2 b_{34}$ and $\mathrm{Sq}_*^8 b_{34}$. Here (42) implies that

$$\begin{aligned} y_6 * b_{34} &= y_6 * (y_{18} * b_{16} + b_8 b_{26} + b_4 b_8 b_{22} + b_2 b_4 b_8 b_{10}^2) \\ &= y_{18} * (y_6 * b_{16}) + y_6 * (b_8 b_{26} + b_4 b_8 b_{22} + b_2 b_4 b_8 b_{10}^2) \\ &= b_{10}^4. \end{aligned}$$

By the similar manner we can compute $y_{10} * b_{34}$ and $y_{18} * b_{34}$ as

$$y_{10} * b_{34} = b_{22}^2,$$

$$y_{18} * b_{34} = b_{26}^2. \tag{47}$$

Also by applying Sq_*^8 to (47), we have

$$y_{18} * (\mathrm{Sq}_*^8 b_{34}) + y_{10} * b_{34} = \mathrm{Sq}_*^8(b_{26}^2).$$

This means $y_{18} * (\mathrm{Sq}_*^8 b_{34}) = 0$, while $\mathrm{Sq}_*^8 b_{34} = b_{26}$ or 0. Therefore $\mathrm{Sq}_*^8 b_{34} = 0$.

Also by applying Sq_*^2 to (42), we have

$$\begin{aligned}\mathrm{Sq}_*^2 b_{34} &= y_{18} * (\mathrm{Sq}_*^2 b_{16}) + \mathrm{Sq}_*^2(b_8 b_{26} + b_4 b_8 b_{22} + b_2 b_4 b_8 b_{10}^2) \\ &= b_{16}^2.\end{aligned}$$

Thus we obtain the all entries of the tables in Theorem 5.2.

Q.E.D.

6. Homology ring of $LG\,(G)$

As stated in §1, $LG\,(G)$ is isomorphic to the semi-direct product of G and ΩG. Thus the following diagram commutes (See [6].)

$$\begin{array}{ccccc} \Omega G \times G \times \Omega G \times G & \xrightarrow{\omega} & \Omega G \times \Omega G \times G \times G & \xrightarrow{\lambda \times \mu} & \Omega G \times G \\ \downarrow \Phi \times \Phi & & & & \downarrow \Phi \\ LG\,(G) \times LG\,(G) & & \xrightarrow{\lambda'} & & LG\,(G) \end{array}$$

where $\Phi : \Omega G \times G \to LG\,(G)$ is a map defined by $\Phi(l, g)(t) = l(t) \cdot g$ and λ,λ' and μ are the multiplication maps of ΩG, $LG\,(G)$ and G respectively and ω is the composition

$$(1_{\Omega G} \times T \times 1_G) \circ (1_{\Omega G} \times \mathrm{ad} \times 1_G) \circ (1_{\Omega G} \times \Delta_{*G} \times 1_{\Omega G \times G}).$$

And also, Φ is homeomorphism.

Therefore we have the following theorem.

Theorem 6.1. *Let G be a compact, connected, simply connected Lie group and p a prime. Then*

$\mathrm{H}_*(LG\,(G)\,; \mathbf{Z}/p\mathbf{Z}) \cong \mathrm{H}_*(\Omega G\,; \mathbf{Z}/p\mathbf{Z}) \otimes \mathrm{H}_*(G; \mathbf{Z}/p\mathbf{Z})$ *as $\mathbf{Z}/p\mathbf{Z}$ module*

and the multiplication is defined by

$$(b \otimes y) \cdot (b' \otimes y') = (b \cdot (y_{(2)} * b')) \otimes (y_{(1)} \cdot y')$$

where $b, b' \in \mathrm{H}_*(\Omega G\,; \mathbf{Z}/p\mathbf{Z})$, $y, y' \in \mathrm{H}_*(G; \mathbf{Z}/p\mathbf{Z})$ and $\Delta_* y = \sum y_{(1)} \otimes y_{(2)}$.

Thus by Theorem 4.1, 4.4 and 5.2 we can directly compute the algebra structure of $\mathrm{H}_*(LG\,(G)\,; \mathbf{Z}/2\mathbf{Z})$ for $G = G_2, F_4, E_6, E_7$. But it is complex to write them out exactly. Hence we show the case of G_2 only.

Theorem 6.2. $\mathrm{H}_*(LG\,(G_2)\,; \mathbf{Z}/2\mathbf{Z})$ *is generated by y_3, y_5, y_6 and b_2, b_4, b_{10}. And their fundamental relations are*

$$y_3^2 = 0,\ y_5^2 = 0,\ y_6^2 = 0,\ b_2^2 = 0,$$

$$[y_i, y_j] = 0,\ [b_i, b_j] = 0\ [y_3, b_i] = 0,\ [y_5, b_i] = 0,$$

$$[y_6, b_2] = b_4^2,\ [y_6, b_4] = b_{10},\ [y_6, b_{10}] = b_4^4.$$

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