A memo about the definition of projective n-spaces of a Hopf space.

Hiroaki Hamanaka

In [1], the projective i-space $XP(i)$ of an associative H-space X is defined as follows.

Definition 0.0.1. Let X be an associative H-space. Then $XP(n)$ is defined inductively as follows.

Let Δ^n be the standard n-simplex and $\delta_i : \Delta^{n-1} \to \Delta^n$ $(0 \leq i \leq n)$ and $s_i : \Delta^n \to \Delta^{n-1}$ $(1 \leq i \leq n)$ be the face and the degeneracy map, respectively.

The following map is claimed to be a relative homeomorphism.

$$\zeta_n : (\Delta^n \times X^n, \delta\Delta^n \times X^n \cup \Delta^n \times X^{[n]}) \to (XP(n), XP(n-1)),$$

where ζ_n is defined by

$$\zeta_n(\delta_i(\sigma), x_1, ..., x_n) = \begin{cases} \zeta_{n-1}(\sigma, x_2, ..., x_n) & i = 0, \\ \zeta_{n-1}(\sigma, x_1, ..., x_{n-1}) & i = n, \\ \zeta_{n-1}(\sigma, x_1, ., x_i \cdot x_{i+1}, .., x_n) & 0 < i < n, \end{cases}$$

and

$$\zeta_n(\sigma, x_1, ..., x_n) = \zeta_{n-1}(s_j(\sigma), x_1, ..., \hat{x_j}, ..., x_n)$$

when $x_j = *$.

Here $X^{[n]} = \{(x_1, ..., x_n) \in X^n |$ for some j $x_j = *\}$.

This definition may be somewhat difficult for beginners. First you should observe that the initial condition of the inductive definition, that is, $XP(0) = *$, since the domain of ζ_0 is just $(*, \emptyset)$.

Next you will consider $XP(1)$. What is $XP(1)$? The above definition says that

$$\zeta_1 : ([0,1] \times X, \{0,1\} \times X \cup [0,1] \times *) \to (XP(1), *)$$

is a relative homeomorphism. Yes, $XP(1)$ is just the suspension space of X. But we consider as following. $XP(1)$ is a quotient space of $\{(t_0, x, t_1) \in [0,1] \times X \times [0,1] | t_0 + t_1 = 1\}$ with the equivalence relation:

$$(0, x, 1) \sim (t_0, *, t_1) \sim (1, x, 0).$$

The similar consideration goes for $n \geq 2$. Now we can give the substitutive definition of $XP(n)$ which may be more comprehensible:

Definition 0.0.2. Let X be an associative H-space. We call

$$\{(t_0, x_1, t_1, x_2, t_2, ..., x_n, t_n) | t_i \in [0,1], x_i \in X, \sum t_i = 1\}$$

as weighted product $X^{w(n)}$ of X. Then $XP(n)$ is the quotient space of $\bigcup_{0 \leq i \leq n} X^{w(n)}$ and the equivalence relation

$$\begin{aligned} (0, x_1, t_1, x_2, t_2, ..., x_n, t_n) &\sim (t_1, x_2, t_2, ..., x_n, t_n) \\ (t_0, x_1, t_1, x_2, t_2, ...t_{n-1}, x_n, 0) &\sim (t_0, x_1, t_1, ..., t_{n-1}) \\ (t_0, x_1, t_1, ...x_i, 0, x_{i+1}, ..., x_n, t_n) &\sim (t_0, x_1, t_1, ..., x_i \cdot x_{i+1}, ..., x_n, t_n) \\ (t_0, x_1, t_1, ...t_{i-1}, *, t_i, ..., x_n, t_n) &\sim (t_0, x_1, t_1, ..., t_{i-1} + t_i, ..., x_n, t_n) \end{aligned}$$

[1] Yutaka Hemmi,*Higher homotopy commutativity of H-spaces and the mod p torus theorem* Pacific J. Math. vol.149, No.1,(1991), 95–111.